**Every planar graph without adjacent triangles or 7-cycles is - choosable**

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**Abstract**

In a graph , a list assignment  is a function that it assigns a list of colors to each vertex  An coloring is a mapping  that assigns a color  to each vertex  so that at most impropriety  neighbors of are the same color with . A graph  is said to be  choosable if it admits an coloring for every list assignment  with  for all . In this paper, we prove that every planar graph with neither adjacent triangles nor 7-cycles is choosable. In 2016, Min Chen, Andre Raspaud and Weifan Wang proved that every planar graph with neither adjacent triangles nor 6-cycles is choosable.

Keywords: Planar graphs, improper choosability, cycle.

1. **Introduction**

A  coloring of  is a mapping  from  to a color set such that for any adjacent vertices  and  A graph is if it has a  Cowen et al.(1986) considered defective coloring of graphs. A graph  is said to be   or simply,  if the vertices of  can be colored with  colors in such a way that vertex has at most  neighbors receiving the same color as itself. Clearly, a  is an ordinary proper 

A list assignment of  is a function  that assigns a list  of colors to each vertex  An  with impropriety of integer  or simply an  of  is a mapping  that assigns a color  to each vertex  so that at most neighbors of  receive color  A graph is  with impropriety of integer  or simply  if there exists an  for every is just the ordinary choosability introduced by Erdös et al. (1979) and independently by Vizing (1976). A famous and classic result given by Thomassen (1994) is that every planar graph is choosable. However, Voigt (1993) showed that not all planar graphs are choosable by establishing a nonchoosable planar graph.

In 1999, Šrekovski(1999a) and Eaton and Hull (1999) independently introduced the concept of list improper coloring. They showed that planar graphs are choosable and outerplanar graphs are choosable. They are both improvement of the results shown in Cowen et al. (1986) which say that planar graphs are colorable and outerplanar graphs are colorable. Note that there exist noncolorable planar graphs and noncolorable outerplanar graphs which were constructed in Cowen et al. (1986). Let  denote the girth of a graph  i.e., the length of a shortest cycle in  The choosability of planar graph with given  has been investigated by Šrekovski (2000). He proved that every planar graph  is  choosable if  choosable if  choosable if  and  choosable if and  The first two results were strengthened by Havet and Sereni (2006) who proved that every planar graph  is  choosable if  and  choosable if  Recently, Cushing and Kierstead (2010) proved that every planar graph is  choosable. So it would be interesting to investigate the sufficient conditions of  choosability of subfamilies of planar graphs where some families of cycles are forbidden. Šrekovski proved in Šrekovski (1999b) that every planar graph without 3-cycles is  choosable. Lih et al. (2001) proved that planar graphs without 4- and cycles are choosable, where  Later, Dong and Xu (2009) proved that planar graphs without 4- and  cycles are  choosable, where  These two results were improved further by Wang and Xu (2013) who showed that every planar graph without 4-cycles is  choosable. More recently, Chen and Raspaud (2014) proved that every planar with neither adjacent 4-cycles nor 4-cycles adjacent to 3-cycles is  choosable. This absorbs above results in Lih et al. (2001), Dong and Xu (2009), Wang and Xu (2013). Then, Min Chen, Andre Raspaud and Weifan Wang (2016) proved that every planar graph with neither adjacent triangles nor 6-cycles is  choosable.

**Theorem 1.1** Every planar graph with neither adjacent triangles nor 7-cycles is  choosable.

The proof of Theorem 1.1 is done in the section 3.

**2 Notation**

All graphs considered in this paper are finite, simple and undirected without multiple edges. Call a graph  planar if it can be embedded into the plane so that its edges meet only at their ends. Any such particular embedding of a planar graph is called a plane graph. For a plane graph  we use  and   to denote its vertex set, edge set, face set, maximum degree and minimum degree, respectively. For a vertex  the degree of  in , denoted by  or simply  is the number of edges incident with  in .  and  are order and size. The neighborhood of  in  denoted by  or simply  consists of all vertices adjacent to  in  Call  a vertex, or a  vertex, or a  vertex if  or  or  respectively. A similar notation will be used for cycles and faces. For a face  the number of edges of the boundary of  (where cut edge, if any, is counted twice), denoted by , is called the degree of . Analogously, the notations above for vertices will be applied to faces. We write  if  are consecutive vertices on  in a cyclic order, and say that  is a face. Next, let  be the face with  and  as two boundary edges for  where indices are taken modulo  and define  Let  be a vertex, and  is a 3-vertex in  such that the three neighbors vertices adjacent with  An edge  is called a edge, and  is called a neighbor of  A cycle is a cycle of length  In this paper, a 3-face is often called a triangle. Call a vertex or an edge triangular if it is incident with a triangle. Otherwise, a vertex or an edge **iso-triangular** if it is not incident with a triangle but its neighbor vertex is incident with triangle. Then 4-face is often called a quadrilateral. Two cycles or two faces are intersecting if they have at least one vertex in common; and are adjacent if they have at least one edge in common. Again, 4-face is called a quadrilateral in which two triangles are adjacent.

We define the following notation:

* Let  be a 4-vertex. If  is incident with    and  so that face and then  and face. It is called **4-light vertex**. Shown in Figure 1.





Figure 1:

**Definition 2.1** Let  be 3-face such that and be an edge incident with 

i.e.,    can be written by 

**Definition 2.2**

* A 3-vertex is said to be **poor** if it is incident with one 3-face and two 4-faces. Then it is called **3-poor.**
* Let  be a 4-vertex and  be a 3-face. If  is incident with one 3-face, one 4-face and one 5-face adjacent with  and another is 6-face, then it is said to be 4**-poor**. (OR)
* A 4-vertex is said to be **poor** if it is incident with one 3-face and two of  incident with one 4-face and one 5-face and another is 6-face. Then it is called **4-poor**.
* Let  be a 5-vertex and  be a 3-face. If  is incident with one 3-face and both one 4-face and one 5-face adjacent with  and others' two are face and face, then it is said to be **5-poor**.

(OR)

A 5-vertex is said to be **poor** if it is incident with one 3-face and two of  incident with one 4-face and one 5-face and others are incident with  face and face. Then it is called **5-poor**.

**Definition 2.3**

* A 3-vertex is said to be **semi-poor** if it is incident with three 4-faces. Then it is called **3-semi-poor**.
* A 4-vertex is said to be **semi-poor** if it is incident with one 3-face adjacent to one 4-face and one 4-face adjacent to one 3-face. Then it is also called a **semi-poor-I** vertex.
* A 4-vertex is said to be **semi-poor** if it is incident with one 3-face adjacent to one 4-face and one 4-face adjacent to one 4-face. Then it is also called a **semi-poor-II** vertex.
* A 4-vertex is said to be **semi-poor** if it is incident with one 3-face adjacent to one 5-face and one 4-face adjacent to one 3-face. Then it is also called a **semi-poor-III** vertex.
* A 4-vertex is said to be **semi-poor** if it is incident with one 3-face adjacent to one 5-face and one 4-face adjacent to one 4-face. Then it is also called a **semi-poor-IV** vertex.

**Definition 2.4**

* A 3-vertex is said to be **full-poor** if it is incident with one 3-face, one 5-face and face. Then it is called **3-full-poor**.
* A 4-vertex is said to be **full-poor** if it is incident with one 4-face adjacent to one 3-face and one 4-face adjacent to one 3-face. Then it is also called a **full-poor-I** vertex.
* A 4-vertex is said to be **full-poor** if it is incident with one 4-face adjacent to one 3-face and one 4-face adjacent to one 4-face. Then it is also called a **full-poor-II** vertex.
* A 4-vertex is said to be **full-poor** if it is incident with one 4-face adjacent to one 4-face and one 4-face adjacent to one 4-face. Then it is also called a **full-poor-III** vertex.

Figure 2:

3-poor

3-semi poor



3-full poor

Figure 3:

4-poor

4-semi poor I

4-semi poor II

Figure 4:

4-semi poor III

4-semi poor IV

Figure 5:

4-full poor I

4-full poor III

4-full poor II

Figure 6:

5-poor

-face

-face

**Theorem 2.5 (Chen [1]).** Every planar graph neither adjacent triangle nor 6 cycle is choosable.

**Theorem 2.6 (Chen [2]).** Every planar graph without 4-cycles adjacent to 3- and 4-cycles is choosable.

**Lemma 2.7 (Lih, Wang, Zhang [9]).**

(1) 

(2) No two adjacent 3-vertices.

**Lemma 2.8** Let  be (3,4,5)-face. Then all vertices of  are poor.

**Proof:** Let face and then   and  Suppose to the contrary that there is no poor vertex of in  Let  By minimality of  suppose that  has an coloring of 

First, for  without loss of generality, let be a quadrilateral and  be not incident with 4-face. We may provide the colors and We must have the color with  So, we choose the color with 3. If we recolor  with  then we will get the color of the same If we recolor  with 3, we can exchange the colors  and However, since is not incident with 4-face, it means that it is incident with 8-face. So,  and can be adjacent to each other. If  is a triangle, we must have the color  with 3. So, it is impossible for the color  with 3. If  is not a triangle,  can be a triangle. So, we can assume that the colors and with 3. Since  is not incident with 4-face, so  So, we could have the colors and  are the same. Then we change the colors and  It is contradiction for  vertex.

Secondly, for  and  we have proved that  is a poor vertex. Without loss of generality, we have  and  are quadrilaterals and then we cannot have both  is a triangle and  is a quadrilateral. So, we may assume that  is a triangle. Since  is not incident with 4-,5-,6-faces. Without loss of generality, let    and  If we provide the colors   and  then we must have the colors  with 4 and  with 4. We can give the color  with . If we recolor  with 4, we must exchange the colors  and  However,  It is impossible. Thus, it is contradiction by assumption. Therefore, the proof is complete.

**Lemma 2.9** If  be a (4,4,4,4)-face, then every vertex of 4-face can be a 4-light vertex.

**Proof:** Let be a 4-face in which every vertex is a 4-vertex. Assume that   and are the neighbors of  composing of a triangle with their neighbors where Suppose to the contrary that none of  is a 4-light vertex such that  where Let   By the minimality of  admits an  coloring of  We will consider two cases.

Case (i) We may give colors with  and  are the same and  and  are also. So, let and  Thus, we can deduce that and  where  and   We consider three sub-cases in the following.

Sub-case (i) Firstly, for  we will consider and have to be incident with only one triangle. By assumption, we have face. We must have the colors  If  is a quadrilateral, we cannot give the same colors

  and  So, we may assume that    Here, we must have the colors  If we exchange the colors  and  we must recolor  with 2 or 3. Clearly,  is impossible. So, we must have the color with 3. Moreover, secondly, for the vertex we will consider and have to be incident with only one triangle. We may assume that   If  is a quadrilateral, we have different colors between and So, if we assume that we must have the colors with 3. Clearly, we have and If we exchange the colors and  We must recolor with 3. It is contradiction by assumption.

Sub-case (ii) For the vertex we will consider and have to be incident with triangle. We must have the colors  Let be a triangle and be a quadrilateral. We may assume that   Here, we must have the color  If we exchange the colors  and and then the colors  and we must recolor  with 3. Moreover, for the vertex we will consider  and have to be incident with triangle. Let be a triangle and be a quadrilateral. We may assume that  and  So, we must have the color If we exchange the colors and  it is impossible for  Thus, we will exchange the colors and . It is contradiction by assumption.

Sub-case (iii) For the vertex  we will consider  and to be incident with three triangles. Obviously,  and do not be incident with any quadrilateral. Let and  We must have the colors  with 3 and  with 1. Similarly, we will consider the vertex Let and  We must obtain the colors with 3 and  with 2. If we recolor any vertex, it is very strict. Since  and where  are incident with only face, any neighbor of   and and any neighbor of and cannot be adjacent to each other. Here, coloring is satisfied. Thus, it is contradiction. It is enough to prove only two vertices  and .

Case(ii) We may give colors with  and are different. So, let and and and We must have the colors  and where  Suppose that  We must have  If we exchange the colors and  we must have colors  If we have the colors  with 3, it is impossible because of  So, there is the color  with 2. If we exchange the colors and  we must have colors  If we have a color  with 2, it is impossible. So, there must be the color  with 1. If we exchange the colors  and we must have colors  It is impossible for two of  So, we must recolor the colors with  Thus, it is contradiction for suggestion.

Similarly, for the vertex  and  we can deduce that the resulting coloring is an coloring, which is a contradiction. Therefore, the proof is complete.

**Lemma 2.10** Let  be a 3-face by face.

1. If 3-vertex is a 3-poor vertex, then none of two 4-vertices is a 4-semi-poor vertex.
2. If a 3-vertex is a 3-poor vertex, then the neighbors of the third vertex not on  is vertices.
3. If a 3-vertex is a 3-poor vertex, then at most one vertex of the neighbors of two 4-vertices is 3-vertex.

**Proof:** Let face and  and where  We will prove the first  Let  be a 3-poor vertex. Suppose to the contrary that  is a 4-semi-poor vertex in which  We note that  has a 4-vertex incident  and  and then  is incident with  Let  By minimality of  suppose that has an coloring of Without loss of generality, let   and  Since so we can assign the color  with 2 or 3. If we recolor  with 2, then we must assign the color with 1. But  So, we must assign the color with 2 or 3. Here, by assumption,  must be a quadrilateral. So,  must be 2. Hence we must assign the color  with 3. If we choose the colors with 3 and  with 2, we must assign the color  with 2. If we choose the colors with 2 and  with 3, then we must assign the color with 3. If we recolor  with 3, then we must assign the color with 2 or 1. If we choose with 2 and with 1, then we must assign the color  with 1 or 3 and with 2 or 3. If we choose the color with 3, then we must assign the color with 2. Thus, it is contradiction by assumption. If we choose the color  with 1, then we must assign the colors with 3 and  with 2. If we choose the colors  with 3 and  with 3, then it is contradiction by assumption. If we choose the color  with 2 and  with 3, then it is contradiction.

We will prove the second  and  simultaneously. Here, since  is incident with two 4-faces by Theorem 1.1 , so cannot be incident with any 4-faces. Thus, we have to know that it could be incident with faces. So, and  However,  and cannot be adjacent to 3-vertex because of  and  are not 4-poor vertices. Therefore, the proof is complete.

**Lemma 2.11** Let  be a 3-vertex in a graph . If  is a 3-semi poor vertex, then none of 4-face incident with  can be adjacent to

1. a 4-poor vertex,
2. a 4-semi poor I vertex and
3. a 4-semi poor III vertex.

**Proof:** Let  be a 3-semi poor vertex in a graph  and  and  and then  We will prove first condition **(i)**. Suppose to the contrary that all of   and  are incident with 4-poor vertex. Firstly, we will prove a 4-poor vertex incident with  and . Without loss of generality, suppose that all of   and are incident with a 4-poor vertex. Here, obviously we will assume that each of   and is incident with a 4-poor vertex. We will consider  vertex by contraction of   and  So, let Continuously, we may construct each triangle incident with  such as  and Then  is incident with both 5-face and 6-face. Let  By minimality of  suppose that  has an coloring of  We will consider two cases.

Case  We may assume that  and are the same colors and  and are the same. So, we may assign the colors   and with 1 and then the colors  and  with 2. Here, we must assign the color  with  and we must assign the color  with 3. Evidently, 5-face is 3-coloring and 6-face is 2-coloring. So, we must assign the colors  with 1. Here, we will assign the color with 3. Here, we must have all colors   and  with 2. If we exchange the colors  and we must recolor with  with  and  with  Since it must be  Now, we can have the color  with 2. It is contradiction. Moreover, since  and  are incident with 6-face and we have that 6-face is 2-coloring, they must be the colors and  with 2. So, we must have the colors  and  with 3. It is contradiction.

Furthermore, since  we must assign the color  with 2. If we exchange the colors  and  we must recolor with  and  with  So, we must have the colors  and with 3. Then, we will exchange the colors  and  However, it is contradiction by assumption.

Case  We may assume that   and  are different. Evidently, we must have the colors  and  are different. We may assume that the colors with 1,  with 2 and with 3. So, we must have the colors with 3,  with 1 and  with 2 and then continuously we must have the colors  with 2,  with 3 and  with 1. If we assign the color  with 1, then we must recolor with  Thus, we must have the color  with distinct  Here, it is contradiction.

If we assign the color  with 2, then we must recolor with  Here, we must have the color  with distinct  However, it is contradiction. If we assign the color with 3, then we must recolor  with  Here, we must have the color  with distinct  However, it is contradiction.

Finally, for the condition and  are similar as the proof of the condition  Therefore, the proof is complete.

**Corollary 2.12** Suppose to is a 3-semi-poor vertex in which  and  If the three vertices of  and  are 3-semi-poor vertices, then the three vertices of   and  are vertices.

**Lemma 2.13** Let  be 3-vertex,  and  If  is a 3-full-poor vertex in which  and  are incident with 5-face, then

1. the three neighbors of  are vertices (i.e.,  and
2. exactly the vertex  is either a 4-poor vertex or a 5-poor vertex.

**Definition 2.14** (i) A vertex  is a vertex incident with at most  triangles and others are any faces. Its vertex is called vertex.

Here, the number of triangles incident with a vertex

(ii) A vertex  is vertex with  in which  is incident with exactly  3-faces and exactly 4-faces. It is said to be a vertex. Evidently, if  is odd, then every 4-face must be incident between two 3-faces.

Note that : If  is a 3-vertex incident with one 3-face and one 4-face or one 5-face, then another is one face. It is called vertex.

**Lemma 2.15** Let  be vertex in 

Conditions: (i) If  is vertex (), then it is incident with distinct one 3-face, one 4-face and one face. It is called a special vertex.

The following conditions:

Let  be vertex in  with 

(ii) If  is vertex (), then it is incident with distinct two 3-faces, one 4-face and one face.

(iii) If  is vertex (where ), then it is incident with distinct two 3-faces, one 4-face, and then others are faces.

(iv) For  if  is a vertex and  is odd, then it is incident with at most two faces and others are incident with at most  faces.

1. For  if  is a vertex and  is even, then it is incident with at most faces.













Figure 7:

**Corollary 2.16** If  is a vertex () in which there are incident with at most  3-faces and at most  4-faces, then there are at most two faces and  faces.

**Corollary 2.17** If  is a vertex () in which there are incident with at most  3-faces and at most  4-faces, then there are at most two faces and  faces.

**3 Discharging process**

We now apply a discharging procedure to reach a contradiction. We first define the initial charge function  on the vertices and faces of  by letting  if  and . We note  and  so that we get the initial function  if  and   It follows from Euler's formula  and the relation



so that the total sum of initial function of the vertices and faces is equal to



Since any discharging procedure preserves the total charge of  if we can define suitable discharging rules to change the initial charge function  to the final charge function  on  such that  for all  then



a contradiction completing the proof of Theorem 1.1 when  is 2-connected.

**Proof of Theorem 1.1**

Since is 2-connected,  has no adjacent 3-faces or 7-cycles and  the following Lemma is obvious.

**Lemma 3.1** (i) In  there is no adjacent 3-faces.

(ii) In  there is a 4-face adjacent to at most two 3-faces. Moreover, when a 4-face is adjacent to at least one 3-face, the 4-face can be adjacent to no 4-face except  is a 3-poor vertex.

(iii) In  there is a 4-face adjacent to at least one 4-face.

(iv) In  there is a 5-face adjacent to at most one 3-face and no adjacent to any 4-face.

(v) In  there is no 6-face adjacent to a 3-face.

We will introduce the discharging rules:

R 1. Charge from a face 

R 1.1. If  then  sends  to each incident vertex.

R 1.2. If  then  sends  to each incident vertex.

R 1.3. If  then  sends  to each incident vertex.

R 1.4. If  then  sends  to each incident vertex.

R 2. Charge to a 3-face  where 

R 2.1. Suppose to  is a 4-light vertex.

Let face. Then  gets  from each face and  from 4-face and it sends  to  Then  gets  from face and  from . After that  gets  from face and sends  to 

R 3. Suppose to  is a poor vertex in which  with 

R 3.1. Let  and  be a 3-poor vertex. Then  gets  from each 4-face and  sends  to 

R 3.2. Let  and  be a 4-poor vertex.  gets  from 5-face and  from 6-face and  gets  from 

R 3.3. Let and  be a 5-poor vertex. gets  from 5-face,  from  face and from face and then  gets  from 

R 4. Suppose to  be a 3-semi-poor vertex in which  and with  where 

R 4.1. Let  and  be a 3-semi-poor vertex. Then  gets  from each 4-face.

R 4.2. Let  and they be 3-semi-poor vertices. So,  a vertex where . Then  gets  from each 4-face and  from each vertex and 4-face sends  to other vertices not 3-semi-poor vertices. Moreover,   and  are like as 

R 5. Suppose to  be a 3-full-poor vertex in which  with 

Then  gets  from 5-face and  from face and  sends  to  Moreover, face sends  to other vertices.

R 6. Suppose to  be a 4-semi-poor vertex in which  and  and  are faces with 

R 6.1 Let be a 4-semi-poor I vertex. Then  gets  from  and  from face and it sends  to 

R 6.1.1 For   gets  from   from 4-face and  from face and then  gets  from  and  from face.

R 6.2 Let  be a 4-semi-poor II vertex. Then  gets  from  and  from face and it sends  to 

R 6.2.1 For   gets  from   from 4-face and  from face.

R 6.2.2 For  if the outer neighbor of  is 4-semi-poor vertex, then  gets  from ,  from 4-face and  from face. If the outer neighbor of  is not 4-semi-poor vertex, then  gets  from  and  from 4-face and  from face.

R 6.3 Let  be a 4-semi-poor III vertex. Then  gets  from  and  from face and it sends  to 

R 6.3.1 For   gets  from   from 5-face and  from face and then  gets  from  and  from face.

R 6.4 Let  be a 4-semi-poor IV vertex. Then gets  from  and  from face and it sends  to 

R 6.4.1 For   gets  from   from 5-face and  from  face.

R 6.4.2 For  if the outer neighbor of  is 4-semi-poor vertex, then gets  from  from 4-face and  from face. If the outer neighbor of  is not 4-semi-poor vertex, then  gets  from  and  from 4-face and  from face.

R 7. Suppose to  be a 4-full-poor vertex in which  and  and are faces with 

R 7.1 Let  be a 4-full-poor I vertex. Then  gets  from each face and it sends  to and 

R 7.1.1 For both  and  get  from 4-face and  from face and then they get  from  Moreover,  and  send  to 3-vertex and  to vertex.

R 7.2 Let  be a 4-full-poor II vertex and  is incident with 3-face and  is incident with 4-face. Then  gets  from each  face and it sends  to and  to 

R 7.2.1 For   gets  from 4-face and  from face and then it gets  from 

R 7.2.2 For  if the outer neighbor of  is 4-semi-poor vertex, then  gets  from   from 4-face and  from  face and then gets  from  If the outer neighbor of  is not 4-semi-poor vertex, then gets  from  and  from 4-face and  from face and then  from 

R 7.3 Let  be a 4-full-poor III vertex. Then  gets  from each face and it sends  to both and 

R 7.3.1 For  if the outer neighbors of  and  is 4-semi-poor vertices, then both of  and  get 1 from each 4-face and  from face and then get  from If the outer neighbors of and  are not 4-semi-poor vertices, then and  get 1 from  and  and  from 4-face and  from  face and then  from 

R 8. Suppose to is vertex.

We deduce induction for 

R 8.1 vertex.

Let  and  be 3-vertex incident with 4-face and face. If is a  vertex, then  gets  from face and  from 4-face. Then  sends  to 

R 8.2 vertex.

If is vertex incident with one 4-face and one face, then  gets  from face and  from 4-face and then  sends  to each 3-face.

R 8.3. vertex

Let and  gets  from each face and  from 4-face. Then  sends  to each 3-face.

R 8.4. vertex

R 8.4.1 Let be a vertex such that  is even and   gets from each face and  from 4-face. In general ,sends  to each 3-face.

R 8.4.2 Let  be a vertex such that  is odd and  Here is incident with  face where  and  and incident with  3-face and two face.

Then gets  from each face,  from 4-face and  from each face.

In general for   and sends  to each 3-face.

R 8.4.3 Let  be a vertex such that  is odd and  Here  is incident with  face where   and  and incident with  3-face and two face.

Then  gets from each face,  from each 4-face and  from each  face.

In general for   and   sends  to 3-face.

R 9. For  if is incident with 3-face, 4-face,  face and face, then  gets  from 4-face,  from face and  from face and sends 1 to 3-face.

R 10. Otherwise, if  is not a poor vertex in which face, then  gets 1 from 4-vertex and  from 5-vertex and then it sends  to 

It remains to show that the resulting final charge  is satisfied with  for all  Let  and  The proof can be completed with  for all  Let  and  Since  If  by **R 1** and **R 2,** then  is a 4-light vertex with face. So, by **R 2.1**. Continuously, if  by **R 2.1** and **R 5**, then face and the 3-vertex is 3-full-poor vertex. So, by **R 2.1** and  **R 5**.

If by **R 1** and **R 3** and by Lemma 2.8, then ,  and  are 3-poor, 4-poor and 5-poor vertices. So, for  by **R 3.1**. And then for  **R 3.2**. Moreover, for  **R 3.3**. If  and and  then  is a 3-semi-poor vertex by **R 1** and **R 4**. So, we have by **R 4.1**. By Corollary 2.12 if and they are 3-semi-poor vertices, then  So, by **R 4.2**. If  and  and  by **R 1** and **R 5** and by Lemma 2.13, then  is a 3-full-poor vertex. So,  by **R 5**. Then, if  is a 4-poor vertex, then  is incident with 4-face, face and face. So, for  by **R 9** and **R 5**. Here, for 3-face,  **R 3.2** and **R 5** and **R 9**.

For  if  and and are faces with then  is a 4-semi-poor vertex by **R 1** and **R 6**. If  is a 4-semi-poor vertex I, then  by **R 6.1**. For  we must have  So, by **R 6.1.1** and **R 9**. Then, by **R 6.1**, **R 6.1.1** and **R 9**. For  if  is incident with face, then  by **R 6.1.1** and **R 10**. If  is a 4-semi-poor vertex II, then  by **R 6.2**. For  if the outer neighbor of  is 4-semi-poor vertex, then  by **R** **6.2.2**. For  if the outer neighbor of  is 4-full-poor vertex, then  by **R 6.2.2** and **R 7.1**.

If  is a 4-semi-poor vertex III, then by **R 6.3**. For we must have  So,  by **R 6.3.1** and **R 9**. Then, by **R 6.3**, **R 6.3.1** and **R 10**. For  if  is incident with face, then  by **R 6.3.1** and **R 10**.

If  is a 4-semi-poor vertex IV, then  by **R 6.4**. For  if the outer neighbor of  is 4-semi-poor vertex, then  by **R 6.4.2**. For  if the outer neighbor of  is 4-full-poor vertex, then  by **R 6.4.2** and **R 7.1**.

For  if   and  and  are faces with then  is a 4-full-poor vertex by **R 1** and **R 7**. If  is a 4-full-poor vertex I, then  by **R 7.1**. For if and  are incident with, then  by **R 7.1.1** and **R 11** (where  is represented by  and  If  is a 4-full-poor vertex II, then  by **R 7.2**. For  if the outer neighbor of  is 4-semi-poor vertex, then  by **R 7.2.2** and **R 6.1**. For  if the outer neighbor of  is 4-full-poor vertex, then  by **R 7.2.2** and **R 7.1**.

For  by **R 1** and **R 8**, if  is incident with 3-face, 4-face and face, then  is a vertex. Let face. Here,  is vertex and we can get  is a 4-semi-poor vertex and  and so  by **R 8.1**, **R 6** and **R 9**. Then  by **R 8.1**, **R 6** and **R 9**.

For  by **R 1** and **R 8**, if  is incident with two 3-faces, one 4-face and one face, then  is a  vertex. Let  and,  be 4-face and  is face. So,  by **R 8.2**. Let  If  is a vertex, then  by **R 8.2**, **R 10** or  by **R 8.2**, **R 3.1**. So, it is impossible that vertex is adjacent to 3-vertex.

**Lemma 3.2** Let  and ,  be 4-face and  is face. If  is a vertex, then the neighbor vertices of  are vertex.

For  by **R 1** and **R 8**, if  is incident with two 3-faces, one 4-face and two face, then  is a vertex. Let ] and,  be 4-face and  and  are faces. So,  by **R 8.3**. For  and  if  is a vertex, then  by **R 8.3**, **R 9** and **R 3.1** or  by **R 8.2**, **R 3.1** and **R 10**. So, it is impossible that vertex is adjacent to 3-poor vertex. Then  by **R 8.2**, **R 10** and  by **R 8.2, R 9**. Therefore, if  is vertex adjacent to vertex, then face.

**Lemma 3.3**  In  let  be a vertex in which  and ,  be 4-face and  be faces. If a vertex is adjacent to vertex, then face.

Moreover, if  is a vertex, where  and  is even, by Lemma 2.15, then  is incident at most  3-faces, at most  4-faces and at most faces. So, by **R 1** and R 8,



by **R 8.4.1**.

If  is a vertex , where ) by **R 8.4.2** and by Corollary 2.16, then



If  is a vertex (, where) by **R 8.4.3** and by Corollary 2.17, then



If  is a 4-light vertex, then face by **R1** and **R2.1** and **R 5**. If  and are 3-full-poor vertices, then  By Lemma 2.9, when  sends  to each 4-light vertex.  by **R 2.1** and **R 1**. Suppose  with face. By Lemma 2.8 and **R 3**, if   and are poor vertices, then  by **R 3.1**, **R 3.2** and **R 3.3**. By **R 10**, if   and  are not poor vertices, then  For , by Lemma 2.11,  by **R 3.2**, **R 4.1** and **R 6.1**. So, Lemma 2.11 is true.

We have that  is simple, has neither adjacent triangles nor 7-cycles and  the following lemma is obvious. This completes the proof of Theorem 1.1.

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