**Every planar** **graph without adjacent triangles or 7-cycles is (3, 1)∗−choosable**

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**ABSTRACT**

In a graph , a list assignment is a function that it assigns a list of colors to each vertex . An -coloring is a mapping that assigns a color to each vertex so that at most impropriety neighbors of are the same color with . A graph is said to be -choosable if it admits an -coloring for every list assignment with for all . In this paper, we prove that every planar graph with neither adjacent triangles nor 7 -cycles is -choosable. In 2016, Min Chen, Andre Raspaud and Weifan Wang proved that every planar graph with neither adjacent triangles nor 6 -cycles is -choosable.

Keywords: Planar graphs, improper choosability, cycle.

**1.Introduction**

A -coloring of is a mapping from to a color set such that for any adjacent vertices and . A graph is colorabe if it has a -coloring. Cowen et al.(1986) considered defective coloring of graphs. A graph is said to be -improper - colorable, or simply, colorable, if the vertices of can be colored with colors in such a way that vertex has at most neighbors receiving the same color as itself. Clearly, a coloring is an ordinary proper - coloring.  
A list assignment of is a function that assigns a list of col- or to each vertex so that at most neighbors of receive color . A graph is -choosable with impropriety of integer , or simply choosable, if there exists an -coloring for every is just the ordinary -choosability introduced by Erdős et al. (1979) and independently by Vizing (1976). A famous and classic result given by Thomassen (1994) is that every planar graph is -choosable. However, Voigt (1993) showed that not all planar graphs are -choosable by establishing a non- -choosable planar graph.

In 1999, rekovski(1999a) and Eaton and Hull (1999) independently introduced the concept of list improper coloring. They showed that planar graphs are -choosable and outerplanar graphs are -choosable. They are both improvement of the results shown in Cowen et al. (1986) which say that planar graphs are -colorable and outerplanar graph are colorable. Note that there exist non- -colorable planar graphs and non- -colorable outerplanar graphs which were constructed in Cowen et al (1986). Let denote the girth of a graph , i.e., the length of a shortest cycle in . The -choosability of planar graph with given has been investigated by . He proved that every planar graph is -choosable if -choosable if -choosable if , and -choosable if and . The first two results were strengthened by Havet and Sereni (2006) who proved that every planar graph is -choosable if and -choosable if . Recently, Cushing and Kierstesad (2010) proved that every planar graph is -choosable. So it would be interesting to investigate the sufficient conditions of -choosability of subfamilies of planar graphs where some families of cycles are forbidden. Slrekowski prowed in Srekovski (1999b) that every planar graph without 3 -cycles is -choosable. Lih et al.(2001) proved that planar graphs without 4 - and -cycles are -choosable, where . Later, Dong and (2009) proved that planar graphs without 4- and -cycles are -choosable, where . These two results were improved further by Wang and who showed that every planar graph without 4 -cycles is -choosable. More recently, Chen and Raspaud (2014) proved that every planar with neither adjacent 4 -cycles nor 4 -cycles adjacent to 3-cycles is -choosable. This absorbs above results in Lih et al. (2001), Dong and Xu (2009), Wang and Xu (2013). Then, Min Chen, Andre Raspaud and Weifan Wang (2016) prowed that every planar graph with neither adjacent triangles nor 6 -cycles is -choosable.

Theorem 1.1 Every planar gruph with neither adjacent triangles nor 7. cycles is -choosable.

The proof of Theorem 1.1 is done in the section 3.

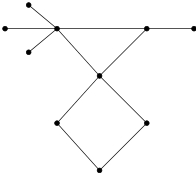
2 Notation

All graphs considered in this paper are finite, simple and undirected without multiple exges. Call a graph planar if it can be embedded into the plane so that its edges meet only at their ends. Any such particular embedding of a planar graph is called a plane graph. For a plane graph , we tise and to denote its vertex set, edge set, face set, maximum degree and minimum degree, respectively. For a vertex , the degree of in , denoted by , or simply , is the number of edges incident with in and are order and size. The neighborhood of in , denoted by , or simply , consists of all vertices adjacent to in . Call a -vertex, or a -vertex, or a -vertex if , or , or , respectively. A similar notation will be used for cycles and faces. For a face ,

the number of edges of the boundary of (where cut edge, if any, is counted twice), denoted by , is called the degree of . Analogously, the notations above for vertices will be applied to faces. We write if are consecutive vertices on in a cyclic order, and say that is a -face. Next, let be the face with and as two boundary edges for , where indices are taken modulo and define . Let be a vertex, and is a in such that the three neighbors vertices adjacent with . An edge is called a -edge, and is called a -neighbor of . A cycle is a cycle of length . In this paper, a 3 -face is often called a triangle. Call a vertex or an edge triangular if it is incident with a triangle. Otherwise, a vertex or an edge iso-triangular if it is not incident with a triangle but its neighbor vertex is incident with triangle. Then 4-face is often called a quadrilateral. Two cycles or two faces are intersecting if they have at least one vertex in common; and are adjacent if they have at least one edge in common. Again, 4-face is called a quadrilateral in which two triangles are adjacent.

We define the following notation:

* Let be a 4 -vertex. If is incident with and so that -face and then and face. It is called a 4-light vertex. Shown in Figure 1.

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8+ − face 8+-face

Figure 1:

Definition 2.1 Let be 3 -face such that and ef be an cdge incident with .  
i.e., can be written by .

Definition 2.2 - A s-verter is said to be poor if it is incident with one 3-face and two 4 -faces. Then it is colled 3 -poor.

* Let be a 4 -vertex and be a 9 -face. If is incident with one 3-fare, one 4-face and one 5-face adjacrnt with ef and another is 6 -face, then it is said to be 4 -poor.  
  (OR)

A 4 -vertex is said to be poor if it is incident with one 3 -face and tuo of incident with one 4 -face and one 5 -face and another is 6 -face. Then it is called 4-poor.

* Let be a 5-vertex and be a 9-face. If is incident with one 3 -face and both one 4 -face and one 5 -face aljacent with and others' two are and , then it is said to be .  
  A 5-vertex is said to be poor if it is incident with one 3-face and tuo of ef incident with one 4-face and one 5-face and others are incident with -face and -face. Then it is called 5 -poor.

Definition 2.3 - A 3 -vertex is suid to be semi-poor if it is incident with three 4 -faces. Then it is called 3 -semi-poor.

* A 4-vertex is said to be semi-poor if it is incident with one 3 -face adjacent to one 4-face and one 4-face adjacent to one $-face. Then it is also called a semi-poor-I verter.
* A 4-vertex is said to be semi-poor if it is incident with one 3 -face adjacent to one 4-face and one 4-face adjacent to one 4-face. Then it is also called a semi-poor-II vertex.
* A f-vertex is said to be semi-poor if it is incident with one 3 -face adjacent to one 5-face and one f-face adjacent to one 9 -face. Then it is also called a semi-poor-III vertex.
* A 4-verter is said to be semi-poor if it is incident with one 3-face adjacent to one 5-face and one 4-face adjacent to one 4-face. Then it is also called a semi-poor-IV vertex.

Definition 2.4 - A S-vertex is said to be full-poor if it is incident with one 3 -face, one 5 -face and -face. Then it is culled 3 -full-poor.

* A 4-verter is said to be full-poor if it is incident with one 4 -face adjacent to one 3-face and one 4-face adjacent to one 3-face. Then it is also called a full-poor-I vertex.
* A f-vertex is said to be full-poor if it is incident with one f-face adjacent to one 3 -face and one 4-face adjacent to one 4-face. Then it is also called a full-poor-II vertex.
* A f-verter is said to be full-poor if it is incident with one 4 -face adjacent to one 4-face and one 4-face adjacent to one 4 -face. Then it is also called a full-poor-III verter.

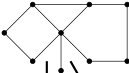
3 -poor 3 Semi -poor 3 -Full poor

Figure: 2





4-poor 4-semi-poor I 4-semi-poor II

Figure: 3



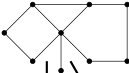
4- Semi Poor III 4 -Semi Poor IV

Figure



4- full Poor 1 4- full Poor II 4- full Poor III

Figure 5:



6+-Face 5+-face

5 Poor

Theorem 2.5 (Chen [1]). Every planar graph neither adjacent triangle nor 6 cycle is -choosable.

Theorem 2.6 (Chen [2]). Every planar gruph without &-cycles adjacent to 3. and 4-cycles is -choosable.

Lemma 2.7 (Lih, Wang, Zhang ).  
(A 1 ) .  
(A 2) No two adjacent s-vertices.

Lemma 2.8 Let be -face. Then all vertices of are poor.  
Proof: Let -face and then and . Suppose to the contrary that there is no poor vertex of in . Let . By minimality of , suppose that has an -coloring of .

First, for , without loss of generality, let be a quadrilateral and be not incident with 4 -face. We may prowide the colors and . We must have the color with . So, we choose the color with 3. If we recolor with , then we will get the color of the same . If we recolor with 3 , we can exchange the colors and . However, since is not incident with 4 -face, it means that it is incident with 8 -face. So, and can be adjacent to each other. If is a triangle, we must have the color with 3 . So, it is impossible for the color with 3 . If is not a triangle, can be a triangle. So, we can assume that the colors and with 3 . Since is not incident with 4 -face, so . So, we could have the colors and are the same. Then we change the colors and . It is contradiction for vertex.

Secondly; for and , we have proved that is a poor vertex. Without loss of generality, we have and are quadrilaterals and then we cannot have both is a triangle and is a quadrilateral. So, we may assume that is a triangle. Since is not incident with -faces. Without loss of generality, let and . If we provide the colors and , then we must have

the colors with 4 and with 4 . We can give the color with - If we recolor with 4 , we matst eschustip: the culces if sal and . Howver, 2 & . It be impocosile. Thass, it is coutradiction iry assumption. Tharebser, the peool is eomplete.

Lemma 2.9 If te at fanc, then evry nerier of 4-fans atn be e 4-fight nerier. that asd ate the neighbors of , compesing of a tristugle with their usighloor whute . Suppoee to the coutrary that wune of -light wertex such that . where . . Let . By the minimality of C. wilnibs an (L. 1)-cobsting of . We will ecestille two casas.

Case (i) We may give colors with and ase the satne atal und abe also. So, let sad . Tlus, we can dethuce that atul , whese and . We coesbber three subt-cicass in the following-

Sub-case (i) Firsaly, fur we will coutwillet sasd hawe to be incident with only case triaugle. By asentuption, we have -lacte. We must have the cilors . If is a quaulrilatital, we counot give the asmat ecobes sad , w may tosatmat that . . Here, we must have the coloes . Ir we exriange the cobses and , we mut trodor with 2 or . Mloriower, secobully, for the writex , we will coctodiler in und yz have to be incibent with only one triauge. We may aseume that . If is a qualtilateral, we have differat colors betworn und . So. if we asoume that , we mast have the colots with 3. CBearly, we hume acs . If we exclange the cobors

Sub-case (ii) Fur the vethex , we will cousilie wad hune to be iracithet with trinagle We mase hane the colun . Lat be an traugle and be a quadrilateral. We may assume that . Here, we muse lume the color . If we esclaage the colots and AN , iod thes the tivers and , we mad recilor with 3. Motowver, for the vertex y. we erill cusbser in and yz howe to be lircident with triangle. Lat yrrive be , we tuit hane the cabe . If exuthuge the cobors and , it is impossible for . Thuse we will excthange the colors sad . It is eoutralsetson by wevuruptson.

quadrilatizal. Let und . We must have the colors with 3 sad with 1 . Similarly, we will coteviller the wetes g. Lut sad . We must obtain the colors and , where , wre incidont with ouly -fwee, uty zavighoe of , prowe anly two vetios and .

Cher(ii) We may give colors vith and are dilleretat. So, let asd sad and . We mutat lave the colkers , und . whuse . Suppese that . We mont have . If we torthatge the culots and , we most have colors . If we huwe the colors with 3 , it is imposible becanse of . So, these ba the colur with 2 . If we earthange the colors wad , wv unat hwve caloes . If h hume a colker if(s) with 2 , it is imposible. Sa. Here mast tee the toblot with 1. Ur we exchutuget the edurs and , we must have mikers the cobots with . Thus, it is ostutrwlietion Lise stuggostion.

Similarly, Fer the vertex and , we can boluce that the resulting coloring is an -coloring, which is a motraliction. Thutusere, the prout is ecmulete.

Lemmia 2.10 Lat be a s-fere by -feore.  
(i) If 3-neriex is a 3-pour verter, then nave of tue f-wcrtions in a f-semipoor verter.  
(ii) a 8 -vertex in a -poor verter, three the neighbors of the third writex not on is -twricis.  
(iii) If e &-tertex is e I-poor wrier, then at wool dove tvertex of the neiyhlors of ture 4 -nerfices in 3 -verter.  
Proof: Lat -fare and and where .

Wer will prowe the lisst (i). Let u be a I-poour vertect. Suppocet to the coutrary that is a 4 -momi-poor vertex in whinh . We rose that his a 4-vester incibent sud romd then be inribent with . Lat hes in -coloring of . Withont lass of giverality, let and . Sinrs , si) we can cowiga the colce with 2 or 3 . If we ticolor with 2 . then we must sosign the colot with 1. But . Sa, we must be s quoulrilateral. So, mimat be 2 . Hemee we must asaign the coler with 3 . If we choose the coloss with 3 und with 2 , we tunst sosign the oblor with 2 .

If we clasuet the colors with 2 and with a, than we most assign the color with 2 of 1 . If we doocedt und with 2 or a. If we choceet the colint with 3 , than we mont we chocose the cober with 1, then we most welign the cobors with 3 und with 2 . If we dhowse that colors with 3 atad with 3. then it is ootrialsetson loy mosumption. If we choose the cobse iM with 2 und with 3, then it is contmalintion. 4-[arso. Thuse, we have to kurw that it cuald be incidoul with farse. So. und . Horwever, dad catunt be haljacout to 3-virtex becsuse of and ase moe 4 -poour vertiose. Thasefore, the prout is coruplete. then nove of 4-fare ricidind with it rus be atjocnt to

(i) e 4-puor wortict.  
(ii) a f-semi poive I terlicx and  
(iii) a f-ncwai poost III twricr. incsbent with 4-poor verter.

Firstly, we will prove a 4-poce vertect incirleat with aul fa. Withonat bose of gowirulit, suppose that all of adil ate incibont with a 4-poor vertex. Here, obvicrsly we will woontme that By minimnlity of , suppose that luct an -ondoring of 3 . We wall cotsaider two civers.

Choe (i). We mov asoture that und ure the sume calors and and ure the sistae. So, we mary asodgn the colorn . sad with 1 and thara the olies und with 2. Were, we must awiga the olor with and we must sosign that cobor with 3. Evibonty, 5-foer in 3-ecibring and fi-fare is 2 -eobsring. So, we mut whign the colors wirh 1. Hete we will sosign that colle with 3 . Here, we mist hawe ull eabes , adal with 2 . If we esoluange the cobors sall , we mat with . Sutace , it must be . Nirw, we cau have the cobor wilh 2. It is contrulieticm. Motowne, since wal with 3. It bo comtruliction.

Further mure, since , we mod asoiga the culor with 2. aul will . Sos we mod have the colors Howerer, it is tamtratiction by asoumpticin.

Case (ii). We may wormat that sal are diffrimat. Evilıutlv. we mast have the colors mal are dillerent. We may cos ume that the colurs with with 2 wall with 3 . So. we munt have the caloes with with 1 and with 2 unt then ootulimasuly we must have the scibss wilh with 3 and with 1. If we asoiga the oolut with 1 , than we mast necbor with Hors, it in coulauliction.

If swiga the cobst with 2 , then we must becalor with Howowt. it is botat rauliction. If we howign tlae colos with 3,1 barn we with distirat . Howevet, it iev countrulictiom. obtiditiom (i). Thavelere, the proot is complete.

Corollary 2.12 Sappose to is a 8-skmi-poior verfex in ataich . semi-poar tertions, Whre tbe there nertios of end is ark -tertion.  
(i) the threx noiglbors of are mertios i.e.. and  
(ii) erarlly the werlex ty in either e f-poor werkex or a 5-poour nerfer.

Definition 2.14 (i) vertex in a -verficr inctilcut with at wobl u-trimigles and others are any foves. Ns merfer in callnd The -verter. Hore, | - the namber of w-triangles focidend wikh a neriex  
(ii) merter is -merfer with dif m miim is inridenf will: craclly 3-faos end exnctiy f-fores. It is said to be e micilont betwoce turo 3-fares.

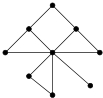
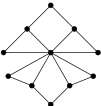
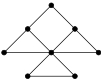
Lemmin 2.15 Lal u be - vertex iv .  
Cubfilima:

3-facs, one 4-face and oue faca. is coliul a spocial -vertex Thes followitiog conditions:  
Lat u be verter in wilh . 3-farrs, one 4 -farx and ane farr. tav S-ferses, wor 4 -fore, and then athers ere farres lent with at most tiro -farcs and athers are incidfot writh at mool -fores. writh at mast faors





T 3 T4 T5

T6 T7

T 6 T 7 T 8

Figure: 7

Figure 7: in which there are incifent wilh af mast S.fares and et mowt 4-farrs, then there are at miast fao -fooss and faose

Corollary is a verfex 4-faras, then there are at mast fimo -feors and -farse

**3 Discharging process**

We soov upply a diathrging peocodure to mact a costrwlistson. We first difias the initial duarge furaction do on the vertions aral fices of lyy let tings, if und . We nute und ios that we get the initial function if and . It followx from Ealer's formula und the relatson

so) that the total sum of initial furction of the wrticis and fucos is equal to

Since any diechurging proosolure preserves the total charge of C. if we can inflise suitahlie discharging rules to clange the initial churge funaction ah to the final charge function of on solh that for all . thin

a coutrudiction comipletling the proof of Thoverm 1.1 when is 2-owntocted.  
Prool of Theorem 1.1 3, the following Lamua bi olwiuts.

Lemma 3.1 (i) In , there is no adjwornt 3-farces.  
(ii) In C, thers is e f-fare efjuront to at muat ture i-faris. Morwoser, whes e f-foor is erjarrat to ef frest ows 3-fars, the f-fare can be adjerant fo soe f-faxe eaript is a -poor worlex.  
(iii) In G, there is a f-farx edjamout to al thest wor f-fers.  
(iv) In G. there is a a-fers arfjarant to et moot ane 3-farr end no affarant to ars 4 -foces.  
(v) In G, there io no 6-fear adjarant to e S-foos.  
We will intecoluce the discharging rulis:

**R 1.** Chatge from a -face   
**R. 1.1.** If , then somal t to ewh incialeat vatex.  
**R. 1.2.** If , then sombe to to ewh incidinat wrtex.

**R. 1.3.** If , then suake fof to ewch incideat votwx.

**R 2.1.** Suppoise to is a 4 -liglat verter.  
Let -five. Then gets from each from - lhee and from . Afer that or gots Irom -fare and suali to .

**R 3.** Suppoest to be a poor vettox in which with .  
**R. 3.1.** Lat and le a 3 -post wortex. Thas gets Irom each 4 -face and f amale to -  
**R. 3.2.** Lat anul be an 4-pose vertex. gds If from 5-frote aul from 6 -fuce and gers from .  
**R. 3.3.** Let und le a 5-poot vertex. vy geb 3 frum 5 -fuct: ftrum Lire und from -face amd then grts ff from 19.

**R 4.** Suppoced to be a 3-armi-poor vortex in which tr wad with whate .  
**Rt 4.1.** Let and = be a 3-ami-poor vertex. Then ges ff frotu each 4 -fince  
**R 4.2.** Let ant thery be 3-somi-poos vetticts. as .

**R 5.** Suppoceet to be a 3-full-poot vertex in which with . Then gis 3 from 5-fuce and from -fact

**R 6.** Suppoceat to be a 4-sumi-poor vertica in which from fioe und it sumb to .  
R E.1.1 For gets + from from 4-fioce face  
**R. 6.2** Lat v lee a 4 -ami-poor II vertix. Then guts from a wall Irotu fhoce ind it sombe to . frum fact.

**R 6.2 .2** For , if the ontes neiglubor of is 4 -semifrom -lace. If the outer neighbor of is not 4 -wemi-poser vertex, then vy gets from und from 4 -lace wad from fact  
**6.3** Lut v be a 4 -semi-poor III wirtex. Then gets ff from ant from cars sud it amals if ta fi-  
**R. 6.3.1** For gits from from 5-fack sual from bare whd thesi ve get from fa aul Ifom fact  
**6.4** Lat be a 4-semi-poor IV vertex. Then gots ff from fa aul Frum tare and it somb af to frum -  
**R 6.4.2** For , if ther onter neiglinor of is poor wertex, then gets if frum fa, ifrom 4-fars sasal if from -fioce. If the outer nighloor of is nut 4-senti-pocer vertex, then vy gets trom fa und &f frum 4 -fiece sad ? from -fare  
uppose to be a 4-full-poor vertios in which vir sayv wad und ave -fines will   
7.1 Lat = be a 4-full-poor I weter. Then ges of from toch fure and it sambla 3 to wall .  
**R. 7.1.1** For , both aud gt from 4-fice und sond to 3 -verter sond to -verters.  
**7.2** Lut e be is 4-full-poor II wetex and in incsbent wirh 3-[act and bs incidond with 4 -lace. Thent gets from ewch -fact and it sombe if to and to to   
**R. 7.2 .** 1 For , mo gets from 4-furs and from -fact sad then it gets from . verlex, them gets lrum of from 4 -foce and of from 4-fice sad of from -fare adal then Erom .  
**7.3** Lat be a 4 -full-poor III vertex. Then gets f from earh - Lace und it sasale ta both and .  
**R. 7.3.1** For , ir the onder nighlors of and is 4-samb-pour vertiovs, thim both of and get 1 from the outer sovighbes of and wre mod 4-bomi-pocer wrticis. then th und get 1 from sad sad &t trom 4 -fare aral ? from Gurs atal then of from

**R 8.** Suppoces to is verter.  
We dediuce induction .  
**R. 8.1.** tvrikx.  
Let and be 3-verterx ifceident with 4-fare hund -fice. If is a vorlex, then gots of from -fice and 1 frum t-face. Thern sotuls 9 to :  
**R. 8.2.** - vxrikx.  
If is -vortex inciblent with one 4 -fuce and ars fuce, thest eart 3-fices.  
**R. 8.3.** wriks  
Let und gets If from inach -fact and if from 4-fwoe. Then samale to to ewch 3-fare.  
**R. 8.**A. wertex  
**R 8.4.1** Lat be a -vertex soch that is even aul . gets 2 from ewh -fince sasd ffrom 4-fure In grastal v  
**R. 8.4.2** Lat be a -vertex such that is call and 7. Hete in incsbont wilh -fauce whare und aul ingident with 3-liute sad two fiest  
Thuse gets of fromi each -fare, of from 4-face and 3 from tath -Sime.  
Lh guaral fot , und sutuls to stach 3-[ave.  
(R 8.4.3) Lat be a -vertex soch that is oald and . Hote is incident wilh fice where and asd incilifot with -fice adal two thes.  
Thim v gets + from varh -fince, trom ench 4 -fice sud ? Iroum earh is Inoe.  
In gexarral foe , und thes gets from 4 -lace, from -face and from -fhoe and amale 1 to 3 -lace:

**R 10.** Oehurwises, ir is rast a pour vetex in whidh -face, thes gess 1 lrom 4 -vertex and frum 5-vertex and them it sunds to . 0) Fer all . Lat sad . The peroal caa be cutupleted

with for . Iot aul . Since . If df(v) sal 2, then vis a 4-light wetex with -fare So, 2.1. Coutinuomly, if by R. 2.1 aud R. 5, thon -face und the 3-wsters is 3-full-poor vettex. by .

If ly R. 1 sad R. 3 sad loy Lumat 2.8, then ly 3.1. And thest fint 3.2. Morevener, fir . R. 3.3 . If and by R 1 uni R 4.5 so, we lauve dh' by R 4.1. By Cocollary 2.12 if atul thery are 3-semipoor vestios, then by R 4.2. If d(v) -3 wad wad by R 1 and R 5 and ly Ievuma 2.13, then in a 3-full-poor wettex. Sos, by R 5. Thun, if is a 4 -poor vertex, ben ive is incilintat with 4-fure, -face and -bare. So, for by . . Here, for Bare, З.2 und acal . with , then is a 4 -bomb-poos vertex hy wad 6. If bi is 4-semi-poor varter I, then by 6.1. For , we mutst have . So, by 6.1.1 and R 9 . Tloen , by R. 6.1. R. 6.1.1 und R. 9. Fir . if is incident with Lars, then by . For , ir the onter usighlour of in 4-ormi-poot vetex, thes by R. 6.2.2. For , if the outer wisflahor of in 4-full-poocr vertioc.

If is is 4 -smai-poor vertex III, then by R 6.3. For , we mist have . So, by 6.3.1 ama R . Then , ly . 6.3, R. 6.3.1 sad R. 10. For . if is incitlost with -Facte, thea ch by R. 6.3 .1 und R 10.

If is a 4 -owmi-poour wertex IV, thaw ly 6.4. For , if the owter nighbor of is 4-acmi-poor vertex, then be by R E.A.2 mad R T.1

Fur , ir und and are -ficts If is a 4 -full-powe vortex 1 , thes

by 7.1. For , if und are insbont with , then by 7.1 .1 mal R 11 (wlare is sepoesituted by und ). If Bs is 4 -[itl-poor wirtex II, thon by R 7.2. For . ar the owter tavighoot of is 4-wami-pose verLiox, then ly R. 7.2 .2 and R 6.1. For , ir the owher nwighbor of is 4-[ull-poser vetwx, then

Fur , In R 1 und R. 8. if Bo inciuluak with Z-fioor, 4-fout Here, is -virtex aul we cau get is a 4 -womi-pose wot wx sud und ly R. 8.1. R. 6 sad R. 9. Thut by R. 8.1. R. 6 and R. 9.  
For , ly und 8, if ins incsbont wirle two 3-farss, was 4-fiwer and une fack, thea v is a vetex. Let ithal be 4-Gare und is fice. by R. 8.2. Lut . Ir is a -vertex, then cli by R 8.2, R. 10 or ly R. 8.2. R. 3.1. So, it is impocsilde that -vertex is mljecent to 3-vortex.

Lemma 3.2 Lat and be 4 -farx erod is und two fioe, than v is a vartex. Lat anal B.3. R 9 iud R. 3.1 ur by R. 8.2 R 3.1 und R 10. So, it is imposbilhle that -vertex is auljuciot to poour vortox. Thus R 8.2, R. 10 and wijuorut to vortex, thas -bare

Lemma 3.3 In , let be a -verlex in which rer and be 4 -fare aud be -fares If a -verlex is efjarout to virtax, born - form.  
Motiver, if is a -vortex, where und is mex, by

by R. 8.4.1.  
If bi a -wetex , whete in 8.4.2 and hy Corcilhary 2.16. tham

8.4.3 mad hy Corcillary 2.17, tham

If o is a 4-light wortex, then -fice by R1 and R2.1 wal . . If und are 3-full-poos wortions, then . By Lumima 29 , when , sunuls to carh 4-light vertex, कh ly R 2.1 und R 1. Suppose by and R 3.3. By R. 10, if wad are hot poos verticts.

then .  
For , by Lemma 2.11, by R. 3.2. R. 4.1 satal R. 6.1. So, Lemma 2.11 is true. und , the following lomina be olwiote. This coupletes the proof of Thasorim 1.I.

**Conclusion:**

Planar graph: A graph that can be embedded in the plane without any edges crossing.

Adjacent triangles or 7-cycles: This means that the graph does not contain any adjacent triangles (cycles of length 3) or 7-cycles (cycles of length 7). In other words, there are no three vertices connected pairwise by edges such that they form a triangle, and there are no cycles of length 7.

(3, 1)-choosable: This refers to a graph coloring property. A graph is said to be (a, b)-choosable if whenever each vertex is assigned a list of at least 'a' colors, and each vertex has at most 'b' neighbors with the same list of colors, then there exists a proper coloring of the graph where each vertex is assigned a color from its list such that no adjacent vertices share the same color.

The conclusion you provided states that every planar graph that does not contain adjacent triangles or 7-cycles is (3, 1)-choosable.

This result likely comes from a deeper proof involving techniques from graph theory and combinatorics. The idea is to show that such graphs can be colored with at most 3 colors in such a way that no adjacent vertices have the same color, given that each vertex has at most 1 neighbor with the same set of available colors.

This kind of result can have applications in various areas, including scheduling problems, network optimization, and other fields where graph coloring plays a role.

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