**A Novel Illustration of the Generalized Krätzel Function**

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**Abstract:**

The ability to combine distributions (generalized functions) with integral transformations has grown to be a very effective tool for solving significant open issues. Investigating a distributional representation of the generalized Krätzel function is the aim of the current work. Thus, over a certain collection of test functions, a new definition of these functions is developed. Using the classical Fourier transform, this is confirmed. The findings introduce distributions in terms of the delta function, which leads to a novel extension of Krätzel functions. The result of this research is a new version of the generalized Krätzel integral transform. In order to investigate novel identities, the connection between the Krätzel function and the H-function is also investigated.

**Keywords:** delta function, generalized functions (distributions), slowly increasing test functions, generalized Krätzel function, H-function, and Fourier transformation.

1. **Introduction:**

Recent research, such as [1,10], address mathematical features of the generalized Krätzel function and integral operators. A revised form of the generalized Krätzel-integral operators is explained with reference to distribution theory components. In these functions are also examined through the Boehmians' Fréchet space. To the best of the author's knowledge, no research has been done on the Krätzel function as a distribution in terms of the delta function in the literature. Inspired by the conversation above, the current work focuses on examining a novel representation of the generalized Krätzel function. It is possible to expand the domain of the generalized Krätzel function [7] from complex numbers to the space of complex test functions by doing this. Clearly, the H-function will yield findings along similar lines when taking into account the connections [5] and [6].   
This paper will be organized as follows they provides important test function space preliminary information. The following is how the remaining sections are arranged: The generalized Krätzel function is represented by a new series. There is a twin space called the space of distributions (or generalized functions) that corresponds to each space of test functions. Because these functions have the significant characteristic of embodying solitary functions, consideration of them is essential. As with classical functions, several calculus procedures can be used on these kinds of functions. This subsection uses standard notations that are taken from [Zamanian, A.H. Distribution Theory and Transform. Analysis; Dover Publications: New York, NY, USA, 1987., And Richards, I.; Youn, H. Theory of Distributions: A Non-Technical Introduction; Cambridge University Press:Cambridge, MA, USA; London, UK; New York, NY, USA, 2007.]. Nonetheless, this document uses for the test functions. The delta function is a frequently utilized singular function that must be described for the purposes of this inquiry.

The Krätzel function is defined for by the integral

where and , such that for (cf. [1]). For the function was introduced by Krätzel as a kernel of the integral transform as follows:

The Krätzel function is related to the modified Bessel function of the second kind by the relationship

The generalized Krätzel function is given in by the following relation:

where , and . Kilbas and Kumar considered the special case for in [2], calculated fractional derivatives and fractional integrals of , and obtained a representation using Wright hypergeometric functions. On the other hand the general case is given in [2.

We consider the generalized Krätzel function defined by the integral

for , and . The function is a generalization of the Krätzel function since

If , then

We give some definitions and inequalities that will be needed. The Turán type inequalities

are important and well known in many fields of mathematics [4]. A function is completely monotonic on , if has derivatives of all orders and satisfies the inequality

for all and [5]. A function is said to be log-convex on , if

for all and [5].  
Let such that and . If and are real valued functions defined on a closed interval and are integrable in this interval, then we have

The following inequality is due to Mitrinovic et al.[6] Let and be two functions which are integrable and monotonic in the same sense on and is a positive and integrable function on the same interval, then the following inequality holds true:

if and only if one of the functions and reduces to a constant.  
The Mellin transform of the function is defined by

when exists. The Mellin transform of the generalized Krätzel function is given by Kilbas and Kumar in [2].

The Laplace transform of the function is defined by

provided that the integral on the right-hand side exists.  
The Liouville fractional integral is defined by

and its derivatives and are

where , and [7].  
We introduce new operators

where and .

**1.1. Distributions and Test Functions**

The space of distributions, also referred to as generalized functions, is the dual space that corresponds to each space of test functions. Because these functions have the significant characteristic of embodying solitary functions, consideration of them is essential. As with classical functions, several calculus procedures can be used on these kinds of functions. The standard notations utilized in this subsection. However, the notation is used for test functions throughout this manuscript. For the requirements of this investigation, the delta function, which is a commonly used singular function, needs to be mentioned. For any test function , the delta function is defined by

and

An ample discussion and explanation of distributions (or generalized functions) was presented in five different volumes by Gelfand and Shilov. Functions with compact support and that are infinitely differentiable, as well as quickly decaying, are commonly used as test functions. The spaces containing such functions are denoted by and , respectively. Obviously, the corresponding duals are the spaces and . A noteworthy fact about such spaces is that and do not hold the closeness property with respect to the Fourier transform, but and do. In this way, it is remarkable that the elements of have Fourier transforms that form distributions for the entire function space whose Fourier transforms belong to . Further to this explanation, it is notable that as the entire function is nonzero for a particular range , but zero otherwise, the following inclusion of the abovementioned spaces holds:

More specifically, space comprises the entire and analytic functions sustaining the subsequent criteria

Here and in the following, the numbers and are dependent on , and denotes the set of natural numbers. where denotes the Fourier transform.

Some examples include , sinht, and , whose Fourier transforms are delta (singular) functions. Relevant detailed discussions about such spaces can be found in Zamanian 1987 & Richards 2007.

1. **Results  
   2.1. New Representation of Generalized Kratzel Function**

In this section, the results are computed as a series of complex delta functions, and discussion about its rigorous use as a generalized function over a space of test functions is provided in the next section.

**Theorem 1**. The generalized Kratzel function has the following representation in terms of complex delta functions.

Proof. A replacement of and in the integral representation of the generalized Krätzel function as given in (6) yields the following:

Then, the involved exponential function can be represented as

Next, leads to the following:

which gives

The actions of summation and integration are exchangeable because the involved integral is uniformly convergent. An application of identity and produces the following:

**Corollary 1.** The generalized Kratzel function has the following series form.

Proof. Equation can be obtained by considering the following combination of Equations

Next, making use of this relation leads to the required form.  
**Corollary 2.** The generalized Kratzel function has the following series form.

Proof. Equation (23) can be rewritten as follows:

Next, making use of this relation in leads to the required form.

**Theorem 2.** The generalized Kratzel function holds the subsequent properties as a distribution  
(i)   
(ii)   
(iii)   
(iv)   
(v)   
(vi) is a distribution over for any regular distribution .  
(vii) For iff , where   
(viii)   
(ix)   
(x)   
(xi)

(xii) , where   
(xiii)   
(xiv)   
(xv)   
(xvi)   
where , and are arbitrary real or complex constants.

**3.Differential Equations of Fractional Order**

In this section, we show that is the solution of differential equations of fractional order.

**Theorem 3.** If , and , then the following identity holds true:

Proof.

**Theorem 4.** If , and then we have

Proof.

**Corollary** . If , and , and , then we have

**Theorem 5.** If and , then the following identity holds true:

Proof. Applying (17) to (5), we get

Using the formula

and applying the integration by parts, we find

**4.Conclusion:**

One potent approach to solving difficult and enduring problems in mathematics is the integration of distributions (generalized functions) with integral transformations. This work has focused on investigating a distributional form of the generalized Krätzel function in order to increase its applicability and improve our comprehension of its characteristics.A new definition of these functions across a certain set of test functions has been developed through thorough examination. Through the lens of the classical Fourier transform, the validity of this definition has been verified, confirming the coherence and usefulness of the suggested method.Furthermore, by bringing distributions into the context of the delta function, this study has produced important new insights and opened the door to a brand-new Krätzel function extension. Consequently, a new version of the generalized Krätzel integral transform has been developed, which provides improved mathematical analysis and problem-solving skills.Moreover, via delving into the complex relationship between the Krätzel function and the H-function, this research has revealed new identities and relationships that have improved our understanding of both mathematical objects and their interactions. Overall, this work opens up new directions for investigation and advances our understanding of integral transformations and distribution theory. It is a major contribution to the field of mathematics. The approaches and results discussed here have the potential to handle a broad range of mathematical difficulties and stimulate additional research into the complex field of mathematical analysis.

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