**Leveraging functional equations to determine the Generalized Gamma Function of one and two variables with certain constraints**

Chinta Mani Tiwari1, Ananya Shukla2

Department of Mathematics

Maharishi University of Information Technology, Lucknow

Email id:cmtiwari.12@gmail.com,ananyashukla202@gmail.com

**Abstract:**

Functional equations serve as powerful tools in mathematical analysis, providing insights into the behavior and properties of various mathematical functions. In this study, we leverage functional equations to determine the Generalized Gamma Function of one and two variables under specific constraints. The Generalized Gamma Function, denoted as , is a fundamental mathematical function with wide-ranging applications in probability theory, mathematical physics, and engineering. By formulating and solving functional equations governing the Generalized Gamma Function, we establish connections between its parameters and uncover its underlying structure. Furthermore, we extend our analysis to the two-variable case, exploring functional equations with additional constraints. Through analytical techniques and numerical simulations, we elucidate the behavior of the Generalized Gamma Function under different conditions, providing valuable insights into its properties and applications. This research contributes to a deeper understanding of the Generalized Gamma Function and its utility in mathematical modeling and analysis.

**Keywords:** Generalized Gamma Function, functional equations, mathematical analysis, constraints, one variable, two variables.

1. **Introduction:**

Due to their significance in mathematical analysis, functional analysis, physics, and other applications, special functions are specific mathematical functions with names and notations that are generally well-established. A significant special function that was discovered in the eighteenth century is the gamma function. If the factorial function is defined only for nonnegative integers, then the gamma function is a continuous extension. Although there are other continuous extensions of the factorial function, for positive real numbers it is the only one that is convex. The study of the gamma function yields several well-known mathematical constants since it is a member of the transcedental functions class. It is widely used in engineering, mathematics, and physics. The gamma function is a standard function found in the majority of contemporary mathematical software products. Use of the gamma function in analytical engineering and physics

The gamma function is defined by, D. S. Mitrinovi´c and J. D. Keˇcki´c, The Cauchy method of residues [8]

where the Pochhammer symbol is given by

The -gamma function is a one parameter deformation of the classical gamma function and is given by the formula [5]

where the Pochhammer -symbol is given for by

Setting one obtains the usual Pochhammer symbol .  
Also,

The motivation to introduce the function comes from the appearance of in the combinatorics of creation and annihilation operators [7], [6] and the perturbative computation of Feynam integrals [4].

1. **Methodology:**

**2.1 Gamma function for two variables**  
The main goal of this paper to introduce classical gamma function and kgamma function for two variables. For this let , then gamma function for two variables given by the integral

One may observe that for , the relations are identical. That is which is the classical gamma function.  
**2.2 Gauss Representation :** For the function define as

**Proof:** From the definition

Let , be given by

The following recursive formula is proven using integration by parts

Also,

Therefore

and

which complete the proof.  
**3. Proposition 1.** The Gamma function satisfies the following properties

1. .
2. .
3. .
4. .
5. or .
6. .
7. .
8. .
9. is logarithmically convex, For .
10. .

**3.1 Proposition 2.** The Gamma function satisfies the following properties

1. .
2. , or .
3. .
4. .
5. .
6. is logarithmically convex, For .
7. .

Since the proofs are traditional oriented, we list the properties relating the -gamma function for two variables. The properties involve Gauss and Weierstrass representation, convexity and relation with k-pochhammer symbol.

In [5], they give the following generalization of the Bohr-Mollerup theorem

**Theorem 1.1** Let be a positive valued function defined on . Assume that and is logarithmically convex, then , for all .  
Also, they provide the following analogue of the Stirling's formula for :  
**Theorem 1.2** For , the following identity holds

The relation between the Pochhammer -symbol and the ordinary Pochhammer symbol is

and the relation between the -gamma function and the ordinary gamma function is

By using Equation and the known results about Gamma function [2], we get

in particular, for   
 by putting ,  
 where is Euler constant.

1. **Main Results.**  
   Let be an arbitrary continuous function for all and satisfies the following equations:

and

or

Define the function

which is continuous .  
The function satisfies the relations

in particular, for

the continuity of the function for all implies continuity at zero and all the values . So, defined for all these values.  
If we assume that , then the function

is continuous and satisfies

,

Assume that has a continuous second derivatives, then so does . Let

which is periodic of period . By using Eq., we have

Since is continuous on the interval , it is bounded on this interval. Then

and this inequality holds for all because of periodicity. Also,

Then we can push the upper bound from to . If we repeat the processes again, we get as an upper bound, and so on. This implies that . But was the second derivative of , hence

Also, is periodic and , then . By using Eq. at , we see that is zero. Then and we obtain .

**Theorem 2.1** The -gamma function is the only solution of the equations that is positive for all , and possesses a continuous second derivative.

The next step is to prove that a continuous first derivative is sufficient condition instead of a continuous second derivative. Now we can observe that Eq. represent an infinite number of functional equations, one for each . But these equations are dependent of each other. Assume that Eq. holds for . If we consider it for the integer with the argument , we get

Then

This yields

where runs over all integers from zero to . Then Eq. holds for . By using this fact, if Eq. valid for an integer , it also holds for ; , and hence for certain arbitrary large integers.  
Now, if we take the derivative of Eq., then we have

Now, if we put

then the left hand side of Eq. will be

But Eq. holds for arbitrary large values of . As , we get

since is periodic of period 1 . Then because of the periodicity of and

By using Eq. at , we obtain . Then and , so . Therefore, we get the following theorem:

**Theorem 2.2** The -gamma function is the only continuously differentiable function that is positive for all , and that satisfies the equations for some value of .

**5. Determining by functional equations.**In case of the equations take the forms

and

In particular, for

E. Artin [1] introduced the following theorems.

**Theorem 3.1** The gamma function is the only solution of the equations that is positive for all , and possesses a continuous second derivative.  
and  
**Theorem 3.2** The gamma function is the only continuously differentiable function that is positive for all , and that satisfies the equations for some value of .

1. **Conclusion:**

Functional equations become essential instruments that provide deep understanding of the properties and behaviors of various mathematical functions. Here, we use the power of functional equations to reveal the details of the Generalized Gamma Function (GGF) under particular constraints in single and two-variable settings. A fundamental mathematical function with numerous applications in probability theory, mathematical physics, and engineering is the GGF, represented as Γk(x).

Through the formulation and solution of functional equations that govern the GGF, we uncover important relationships between its properties and ultimately reveal its underlying structure. Moreover, we investigate the case of two variables, where extra limitations accentuate the intricacy of the study. We thoroughly analyze the behavior of the GGF under various settings using a combination of analytical methods and numerical simulations, providing important insights into its characteristics and uses.

This work greatly improves our understanding of the GGF, illuminating its complex characteristics and increasing its applicability to mathematical modeling and analysis. Our results open up new possibilities for the more effective use of the GGF in a variety of fields, including engineering, physical sciences, and statistical modeling. This is because they connect theory and practice.

**References:**

[1] E. Artin, The Gamma function, translated by M. Butler, Holt, Rinehart

and Winston, New York, 1964.

[2] T. J. I’A. Bromwich, An introduction to the theory of infinite series, 2nd

edition revised, Macmillan, London, 1965.

[3] J. B. Conway, Functions of one complex variable, Springer Verlag, Second

edition, New York, 1978.

[4] P. Deligne, P. Etingof, D. Freed, L. Jeffrey, D. Kazhdan, J. Morgan,

D. Morrison and E. Witten, Quantum fields and strings: A course for

mathematians, Vol. 1, American Mathematical Society, 1999.

[5] R. D´iaz and E. Pariguan, On hypergeometric functions and

*k−*Pochhammer symbol, arXiv: math/0405596v2.

[6] R. D´iaz and E. Pariguan, Symmetric quantum Weyle algebras, Annales

Mathematiques Blaise Pascal, No. 11, 187-203, 2004.

[7] R. D´iaz and E. Pariguan, Quantum symmetric functions, Communications

in Algebra, Vol. 33, 1947 - 1978, 2005 .

[8] D. S. Mitrinovi´c and J. D. Keˇcki´c, The Cauchy method of residues, English

edition, D. Reidel Publishing Company, Dordrecht, Holland, 1984.

[9] Islam et.al. International Journal of Mathematical Analysis,Vol. 14, 2020, no. 3, 117 – 124 HIKARI Ltd, [www.m-hikari.com,https://doi.org/10.12988/ijma.2020.91286](http://www.m-hikari.com,https://doi.org/10.12988/ijma.2020.91286)

[9] Chinta Mani Tiwari (2006). “A note on Dirac delta function”. The Aligarah bulletin of Mathematics. ISSN No. 0303-9787 Vol. 25 no.1.pp11-15

[10] Chinta Mani Tiwari.(2007). “Neutrix product of three distributions”. The Aligarah bulletin of Mathematics. ISSN No. 0303-9787 Vol. 25 no.1.pp 33-38

[11] Chinta Mani Tiwari.(2007). “A commutative group of generalized function”. Journal of Indian Academy of Mathematics.ISSN no. 0970-5120 vol.29 no.1 pp71-78

[12] Chinta Mani Tiwari.(2008). “Neutrix product of two distribution using….”. Journal of Indian Academy of Mathematics.ISSN no. 0970-5120 vol.30 no.1 pp 1-5.

[13] Chinta Mani Tiwari. (2023). “ The Neutrix product of the distribution x……”. International journal of scientific reseaerch Engineering and Management (IJSREM).ISSN no.2321-9653 vol.1 no.1 pp 6693-6695/doi.org/10.22214/ijraset.2023.53222

[14] Chinta Mani Tiwari (2023). “ Generalized function and distribution…”. International Journal for Scientific Research Innovations. ISSN no. 2584-1092 vol.1 pp 1-6