**Every planar graph without adjacent triangles or 7-cycles is (3, 1)∗−choosable**

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**Abstract:**

In a graph G, a list assignment L is a function that it assigns a list L(v) of colors to each vertex v ∈ V (G). An (L, d) ∗−coloring is a mapping β that assigns a color β(v) ∈ L(v) to each vertex v ∈ V (G) so that at most impropriety d neighbors of v are the same color with β(v). A graph G is said to be (k, d) ∗−choosable if it admits an (L, d) ∗−coloring for every list assignment L with |L(v)| ≥ k for all v ∈ V (G). In this paper, we prove that every planar graph with neither adjacent triangles nor 7-cycles is (3, 1)∗−choosable. In 2016, Min Chen, Andre Raspaud and Weifan Wang proved that every planar graph with neither adjacent triangles nor 6-cycles is (3, 1)∗−choosable.

**Keywords:** Planar graphs, improper choosability, cycle.

**1 Introduction**

A k−coloring of G is a mapping β from V (G) to a color set {1, 2, · · · , k} such that β(x) 6= β(y) for any adjacent vertices x and y. A graph is k − colorabe if it has a k − coloring. Cowen et al.(1986) considered defective coloring of graphs. A graph G is said to be d − improper k − colorable, or simply, (k, d) ∗ − colorable, if the vertices of G can be colored with k colors in such a way that vertex has at most d neighbors receiving the same color as itself. Clearly, a (k, 0)∗ − coloring is an ordinary proper k − coloring. A list assignment of G is a function L that assigns a list L(v) of colors to each vertex v ∈ V (G). An L−coloring with impropriety of integer d, or simply an (L, d) ∗ − coloring, of G is a mapping β that assigns a color β(v) ∈ L(v) to each vertex v ∈ V (G) so that at most d neighbors of v receive color β(v). A graph is k − choosable with impropriety of integer d, or simply (k, d) ∗ − choosable, if there exists an (L, d) ∗−coloring for every is just the ordinary k−choosability introduced by Erd˝os et al. (1979) and independently by Vizing (1976). A famous and classic result given by Thomassen (1994) is that every planar graph is (5, 0)∗−choosable. However, Voigt (1993) showed that not all planar graphs are (4, 0)∗−choosable by establishing a non-(4, 0)∗−choosable planar graph. In 1999, Sˇrekovski(1999a) and Eaton and Hull (1999) independently introduced the concept of list improper coloring. They showed that planar graphs are (3, 2)∗−choosable and outerplanar graphs are (2, 2)∗−choosable. They are both improvement of the results shown in Cowen et al. (1986) which say that planar graphs are (3, 2)∗−colorable and outerplanar graphs are (2, 2)∗ colorable. Note that there exist non-(2, 2)∗ -colorable planar graphs and non-(2, 1)∗−colorable outerplanar graphs which were constructed in Cowen et al.(1986). Let g(G) denote the girth of a graph G, i.e., the length of a shortest cycle in G. The (k, d) ∗−choosability of planar graph G with given g(G) has been investigated by Sˇrekovski (2000). He proved that every planar graph G is (2, 1)∗−choosable if g(G) ≥ 9, (2, 2)∗−choosable if g(G) ≥ 7, (2, 3)∗−choosable if g(G) ≥ 6, and (2, d) ∗−choosable if d ≥ 4 and g(G) ≥ 5. The first two results were strengthened by Havet and Sereni (2006) who proved that every planar graph G is (2, 1)∗−choosable if g(G) ≥ 8 and (2, 2)∗−choosable if g(G) ≥ 6. Recently, Cushing and Kierstead (2010) proved that every planar graph is (4, 1)∗−choosable. So it would be interesting to investigate the sufficient conditions of (3, 1)∗−choosability of subfamilies of planar graphs where some families of cycles are forbidden. Sˇlrekovski proved in Sˇrekovski (1999b) that every planar graph without 3-cycles is (3, 1)∗−choosable. Lih et al.(2001) proved that planar graphs without 4- and l−cycles are (3, 1)∗−choosable, where l ∈ {5, 6, 7}. Later, Dong and Xu (2009) proved that planar graphs without 4- and l−cycles are (3, 1)∗−choosable, where l ∈ {8, 9}. These two results were improved further by Wang and Xu (2013) who showed that every planar graph without 4-cycles is (3, 1)∗−choosable. More recently, Chen and Raspaud (2014) proved that every planar with neither adjacent 4-cycles nor 4-cycles adjacent to 3-cycles is (3, 1)∗−choosable. This absorbs above results in Lih et al. (2001), Dong and Xu (2009), Wang and Xu (2013). Then, Min Chen, Andre Raspaud and Weifan Wang (2016) proved that every planar graph with neither adjacent triangles nor 6-cycles is (3, 1)∗−choosable.

Theorem 1.1 Every planar graph with neither adjacent triangles nor 7- cycles is (3, 1)∗−choosable. The proof of Theorem 1.1 is done in the section 3.

**2 Notation**

All graphs considered in this paper are finite, simple and undirected without multiple edges. Call a graph G planar if it can be embedded into the plane so that its edges meet only at their ends. Any such particular embedding of a planar graph is called a plane graph. For a plane graph G, we use V, E, F, ∆ and δ (V (G), E(G), F(G), ∆(G), δ(G)) to denote its vertex set, edge set, face set, maximum degree and minimum degree, respectively. For a vertex v ∈ V, the degree of v in G, denoted by dG(v), or simply d(v), is the number of edges incident with v in G. |V (G)| and |E(G)| are order and size. The neighborhood of v in G, denoted by NG(v), or simply N(v), consists of all vertices adjacent to v in G. Call v a k − vertex, or a k +−vertex, or a k −−vertex if d(v) = k, or d(v) ≥ k, or d(v) ≤ k, respectively. A similar notation will be used for cycles and faces. For a face f ∈ F, the number of edges of the boundary of f (where cut edge, if any, is counted twice), denoted by d(f), is called the degree of f. Analogously, the notations above for vertices will be applied to faces. We write f = [v1v2 · · · vk] if v1, v2, · · · , vk are consecutive vertices on f in a cyclic order, and say that f is a (d(v1), d(v2), · · · , d(vk))−face. Next, let fi be the face with vvi and vvi+1 as two boundary edges for i = 1, 2, · · · , d(v), where indices are taken modulo d(v) and define d(v)+ 1 = 1. Let v be a vertex, and v is a 3−vertex in G such that the three neighbors vertices adjacent with v. An edge xy is called a (d(x), d(y))-edge, and x is called a d(x)−neighbor of y. A k− cycle is a cycle of length k. In this paper, a 3−face is often called a triangle. Call a vertex or an edge triangular if it is incident with a triangle. Otherwise, a vertex or an edge iso-triangular if it is not incident with a triangle but its neighbor vertex is incident with triangle. Then 4-face is often called a quadrilateral. Two cycles or two faces are intersecting if they have at least one vertex in common; and are adjacent if they have at least one edge in common. Again, 4-face is called a quadrilateral in which two triangles are adjacent.

We define the following notation:

• Let u be a 4-vertex. If u is incident with f1, f2, f3 and f4 so that f1 = [uu1u2] = (3, 4, 5 +)−face and then d(f3) = 4 and d(f2) = d(f4) = 8 + − f ace. It is called a 4-light vertex. Shown in Figure 1.

8+ − *face*

8+ − *face*

 Figure 1:

Definition 2.1 Let f be 3-face such that f = [uu1u2] and ef be an edge incident with f.

i.e., euu1 , euu2 , eu1u2 can be written by ef .

Definition 2.2 • A 3-vertex is said to be poor if it is incident with one 3-face and two 4-faces. Then it is called 3-poor. • Let u be a 4-vertex and f = [uu1u2] be a 3-face. If u is incident with one 3-face, one 4-face and one 5-face adjacent with ef and another is 6-face, then it is said to be 4-poor.

(OR)

A 4-vertex is said to be poor if it is incident with one 3-face and two of ef incident with one 4-face and one 5-face and another is 6-face. Then it is called 4-poor.

• Let u be a 5-vertex and f = [uu1u2] be a 3-face. If u is incident with one 3-face and both one 4-face and one 5-face adjacent with ef and others’ two are 6 +−face and 5 +−face, then it is said to be 5-poor.

(OR)

 A 5-vertex is said to be poor if it is incident with one 3-face and two of ef incident with one 4-face and one 5-face and others are incident with 6 +−face and 5 +−face. Then it is called 5-poor. Definition 2.3

 • A 3-vertex is said to be semi-poor if it is incident with three 4-faces. Then it is called 3-semi-poor.

• A 4-vertex is said to be semi-poor if it is incident with one 3-face adjacent to one 4-face and one 4-face adjacent to one 3-face. Then it is also called a semi-poor-I vertex.

 • A 4-vertex is said to be semi-poor if it is incident with one 3-face adjacent to one 4-face and one 4-face adjacent to one 4-face. Then it is also called a semi-poor-II vertex.

• A 4-vertex is said to be semi-poor if it is incident with one 3-face adjacent to one 5-face and one 4-face adjacent to one 3-face. Then it is also called a semi-poor-III vertex.

• A 4-vertex is said to be semi-poor if it is incident with one 3-face adjacent to one 5-face and one 4-face adjacent to one 4-face. Then it is also called a semi-poor-IV vertex. Definition 2.4

• A 3-vertex is said to be full-poor if it is incident with one 3-face, one 5-face and 8 +−face. Then it is called 3-full-poor.

 • A 4-vertex is said to be full-poor if it is incident with one 4-face adjacent to one 3-face and one 4-face adjacent to one 3-face. Then it is also called a full-poor-I vertex.

• A 4-vertex is said to be full-poor if it is incident with one 4-face adjacent to one 3-face and one 4-face adjacent to one 4-face. Then it is also called a full-poor-II vertex.

• A 4-vertex is said to be full-poor if it is incident with one 4-face adjacent to one 4-face and one 4-face adjacent to one 4-face. Then it is also called a full-poor-III vertex.

8+ − *face*

3-poor 3-semi poor

3-full poor

Figure 2:

4-poor 4-semi-poor I4-semi-poor II Figure 3:

4-semi-poor III 4-semi-poor IV Figure 4:

 3-full poor I 4-full-poor II 4-full-poor III Figure 5:

1. poor Figure 6:

6+−face

5+−face

Figure 6: Theorem 2.5 (Chen [1]). Every planar graph neither adjacent triangle nor 6 cycle is (3, 1)∗−choosable.

Theorem 2.6 (Chen [2]). Every planar graph without 4-cycles adjacent to 3- and 4-cycles is (3, 1)∗−choosable.

Lemma 2.7 (Lih, Wang, Zhang [9]). (A 1) δ(G) ≥ 3. (A 2) No two adjacent 3-vertices.

Lemma 2.8 Let f be (3, 4, 5)−face. Then all vertices of f are poor.

Proof: Let f = [xyz] = (3, 4, 5)−face and then x1 ∈ N(x), y1, y2 ∈ N(y) and z1, z2, z3 ∈ N(z). Suppose to the contrary that there is no poor vertex of f in G. Let G0 = {x, y, z, x1, y1, y2, z1, z2, z3}. By minimality of G, suppose that G − G0 has an (L, 1)∗−coloring of β. First, for d(x) = 3, without loss of generality, let xx1y1y be a quadrilateral and exz be not incident with 4−face. We may provide the colors β(y) = β(x1) = β(z1) = 1 and β(y1) = β(z) = 2. We must have the color β(x) with L(x)\ {β(y) S β(z) S β(x1)}. So, we choose the color β(x) with 3. If we recolor β(x1) with L(x1)\{β(y1) S β(x 0 1 )}, then we will get the color of the same β(x). If we recolor β(x1) with 3, we can exchange the colors β(x) and β(z). However, since exz is not incident with 4-face, it means that it is incident with 8-face. So, y1 and x 0 1 can be adjacent to each other. If y1x1x 0 1 is a triangle, we must have the color β(x 0 1 ) with 3. So, it is impossible for the color β(x1) with 3. If y1x1x 0 1 is not a triangle, y1yy2 can be a triangle. So, we can assume that the colors β(x1) and β(y2) with 3. Since exz is not incident with 4-face, so x 0 1 6= z1. So, we could have the colors β(x 0 1 ) and β(z1) are the same. Then we change the colors β(z) and β(z1). It is contradiction for x vertex. Secondly, for d(y) = 4 and d(z) = 5, we have proved that x is a poor vertex. Without loss of generality, we have x1xyy1 and x1xzz1 are quadrilaterals and then we cannot have both yy1y2 is a triangle and yy1 ∗ y2 is a quadrilateral. So, we may assume that zz2z3 is a triangle. Since eyz is not incident with 4-,5-,6-faces. Without loss of generality, let L(x) = L(y1) = L(y2) = L(z1) = {1, 2, 3}, L(y) = L(z2) = {1, 2, 4}, L(z) = L(x1) = {1, 3, 4} and L(z3) = {2, 3, 4}. If we provide the colors β(y1) = β(y2) = β(z2) = 1, β(z1) = 3 and β(y) = β(z3) = 2, then we must have the colors β(x1) with 4 and β(z) with 4. We can give the color β(x) with L(x) \ {β(y) S β(z) S β(x1)}. If we recolor β(y) with 4, we must exchange the colors β(z3) and β(z). However, 2 ∈/ L(z). It is impossible. Thus, it is contradiction by assumption. Therefore, the proof is complete.

Lemma 2.9 If f be a (4,4,4,4)-face, then every vertex of 4-face can be a 4-light vertex.

Proof: Let [xyzw] be a 4-face in which every vertex is a 4-vertex. Assume that xi , yi , zi and wi are the neighbors of x, y, z, w, composing of a triangle with their neighbors where i ∈ {1, 2}. Suppose to the contrary that none of x, y, z, w is a 4-light vertex such that d(Ai) ≥ 4, where Ai = {xi , yi , zi , wi}, i = {1, 2}. Let G0 = {x, y, z, w, xi , yi , zi , wi}, i = {1, 2}. By the minimality of G, G − G0 admits an (L, 1)∗−coloring of β. We will consider two cases. Case (i) We may give colors with β(x) and β(z) are the same and β(y) and β(w) are also. So, let β(x) = β(z) = 1 and β(y) = β(w) = 2. Thus, we can deduce that β(ai) ∈ {2, 3} and β(bi) ∈ {1, 3}, where ai = {xi , zi} and bi = {yi , wi}, i ∈ {1, 2}. We consider three sub-cases in the following. Sub-case (i) Firstly, for x we will consider x1 and x2 have to be incident with only one triangle. By assumption, we have [x1x2x] = (3, 4, 4)−face. We must have the colors {β(x 0 1 ), β(x 0 2 ), β(x 00 2 )} ⊆ {1, 2, 3}. If x1x 0 1x 0 2x2 is a quadrilateral, we cannot give the same colors β(x 0 1 ), β(x 0 2 ) and β(x 00 2 ). So, we may assume that β(x) = β(x 00 2 ) = 1, β(x1) = β(x 0 2 ) = 2, β(x2) = β(x 0 1 ) = 3, β(x 0 1 ) = β(x 00 1 ) = β(x 00 2 ) = 1. Here, we must have the colors β(x 0 2 ) = 2. If we exchange the colors β(x2) and β(x 00 2 ), we must recolor β(x) with 2 or 3. Clearly, β(x) = 2 is impossible. So, we must have the color β(x) with 3. Moreover, secondly, for the vertex y, we will consider y1 and y2 have to be incident with only one triangle. We may assume that β(y1) = 1, β(y2) = 3. If y1y 0 1 y 0 2 y2 is a quadrilateral, we have different colors between y 0 1 and y 0 2 . So, if we assume that β(y 0 2 ) = β(y 00 2 ) = 2, we must have the colors β(y 0 1 ) with 3. Clearly, we have β(y1) = 1 and β(y2) = 3. If we exchange the colors β(y2) and β(y 00 2 ). We must recolor β(y) with 3. It is contradiction by assumption. Sub-case (ii) For the vertex x, we will consider x1 and x2 have to be incident with triangle. We must have the colors {β(x 0 1 ), β(x 0 2 )β(x 00 2 )} ⊆ {1, 2, 3}. Let x2x 0 2x 00 2 be a triangle and x1x 0 1x 0 2x2 be a quadrilateral. We may assume that β(x1) = 2, β(x2) = 3, β(x 0 1 ) = β(x 00 2 ) = 1. Here, we must have the color β(x 0 2 ) = 2. If we exchange the colors β(x1) and β(x 0 1 ), and then the colors β(x2) and β(x 0 2 ), we must recolor β(x) with 3. Moreover, for the vertex y, we will consider y1 and y2 have to be incident with triangle. Let y2y 0 2 y 00 2 be a triangle and y1y 0 1 y 0 2 y2 be a quadrilateral. We may assume that β(y1) = 1, β(y2) = 3. β(y 0 1 ) = β(y 00 2 ) = 2. So, we must have the color β(y 0 2 ) = 1. If we exchange the colors β(y) and β(y1), it is impossible for β(y 0 1 ) ⊆ {1, 3}. Thus, we will exchange the colors β(y) and β(y2). It is contradiction by assumption.

Sub-case (iii) For the vertex x, we will consider x1 and x2 to be incident with three triangles. Obviously, x1 and x2 do not be incident with any quadrilateral. Let β(x1) = β(x 0 2 ) = 2 and β(x 0 1 ) = 3. We must have the colors β(x2) with 3 and β(x 00 2 ) with 1. Similarly, we will consider the vertex y. Let β(y1) = β(y 0 2 ) = 1 and β(y 0 1 ) = 2. We must obtain the colors β(y2) with 3 and β(y 00 2 ) with 2. If we recolor any vertex, it is very strict. Since xi and yi , where i ∈ {1, 2}, are incident with only 8+−face, any neighbor of x 0 1 , x 0 2 and x 00 2 and any neighbor of y 0 1 and y 00 2 cannot be adjacent to each other. Here, (L, 1)−coloring is satisfied. Thus, it is contradiction. It is enough to prove only two vertices x and y.

Case(ii) We may give colors with β(x) and β(y) are different. So, let β(x) = 1 and β(z) = 2 and β(y) = 3 and β(w) = a. We must have the colors β(xi) ∈ {2, 3}, β(yi) ∈ {1, 2}, and β(zi) ∈ {1, 3}, where i ∈ {1, 2}. Suppose that a = 3. We must have β(wi) ∈ {1, 2}. If we exchange the colors β(x) and β(x1), we must have colors β(x) ∈ {2, 3}. If we have the colors β(x) with 3, it is impossible because of β(y) = 3. So, there is the color β(x) with 2. If we exchange the colors β(y) and β(y1), we must have colors β(y) ∈ {1, 2}. If we have a color β(y) with 2, it is impossible. So, there must be the color β(y) with 1. If we exchange the colors β(z) and β(z1), we must have colors β(z) ∈ {1, 3}. It is impossible for two of β(z) ∈ {1, 3}. So, we must recolor the colors β(w) with L(w)\{β(wi) S β(x) S β(z)}. Thus, it is contradiction for suggestion. Similarly, for the vertex z and w, we can deduce that the resulting coloring is an (L, 1)∗−coloring, which is a contradiction. Therefore, the proof is complete.

Lemma 2.10 Let f be a 3-face by (3, 4, 4 +)−face.

 (i) If 3-vertex is a 3-poor vertex, then none of two 4-vertices is a 4-semipoor vertex.

 (ii) If a 3-vertex is a 3-poor vertex, then the neighbors of the third vertex not on ef is 4 +−vertices.

(iii) If a 3-vertex is a 3-poor vertex, then at most one vertex of the neighbors of two 4-vertices is 3-vertex. Proof: Let f = [uu1u2] = (3, 4, 4 +)−face and N(u) = {u1, u2, u3} and N(ui) = {u 0 i , u 00 i } where i = {1, 2}. We will prove the first (i). Let u be a 3-poor vertex. Suppose to the contrary that ui is a 4-semi-poor vertex in which i = {1, 2}. We note that ui has a 4-vertex incident u 0 i and u 00 i and then u 00 i is incident with u3. Let G 0 = {u, u1, u2, u 0 1 , u 00 1 , u 0 2 , u 00 2 , u3}. By minimality of G, suppose that G − G 0 has an (L, 1)∗−coloring of β. Without loss of generality, let β(u) = β(u 00 2 ) = β(u 00 1 ) = 1, β(u1) = β(u 0 2 ) = 2 and β(u2) = β(u 0 1 ) = 3. Since |L(u3)| ≥ 1, so we can assign the color β(u3) with 2 or 3. If we recolor β(u) with 2, then we must assign the color β(u1) with 1. But β(u 00 1 ) = 1. So, we must assign the color β(u 00 1 ) with 2 or 3. Here, by assumption, u1u 0 1 ∗ u 00 1 must be a quadrilateral. So, β(∗) must be 2. Hence we must assign the color β(u 00 1 ) with 3. If we choose the colors β(u 00 1 ) with 3 and β(u3) with 2, we must assign the color β(u 0 1 ) with 2. If we choose the colors β(u 00 1 ) with 2 and β(u3) with 3, then we must assign the color β(u 0 1 ) with 3. If we recolor β(u) with 3, then we must assign the color β(u1) with 2 or 1. If we choose β(u1) with 2 and β(u2) with 1, then we must assign the color β(u 0 1 ) with 1 or 3 and β(u 00 2 ) with 2 or 3. If we choose the color β(u 0 1 ) with 3, then we must assign the color β(u 00 1 ) with 2. Thus, it is contradiction by assumption. If we choose the color β(u 0 1 ) with 1, then we must assign the colors β(u 00 1 ) with 3 and β(u3) with 2. If we choose the colors β(u 00 2 ) with 3 and β(u 0 2 ) with 3, then it is contradiction by assumption. If we choose the color β(u 00 2 ) with 2 and β(u 0 2 ) with 3, then it is contradiction.

We will prove the second (ii) and (iii) simultaneously. Here, since u3 is incident with two 4-faces by Theorem 1.1, so u3 cannot be incident with any 4-faces. Thus, we have to know that it could be incident with 6+− faces. So, d(u 0 3 ) ≥ 4 and d(u 00 1 ) = d(u 00 2 ) = 3. However, u 0 1 and u 00 2 cannot be adjacent to 3-vertex because of u1 and u2 are not 4-poor vertices. Therefore, the proof is complete.

Lemma 2.11 Let u be a 3-vertex in a graph G. If u is a 3-semi poor vertex, then none of 4-face incident with u can be adjacent to

1. a 4-poor vertex,
2. (ii) a 4-semi poor I vertex and
3. (iii) a 4-semi poor III vertex. Proof: Let u be a 3-semi poor vertex in a graph G and f1 = [u1uu2x], f2 = [u2uu3y] and f3 = [u3uu1z] and then N(u) = {u1, u2, u3}. We will prove first condition (i). Suppose to the contrary that all of f1, f2 and f3 are incident with 4-poor vertex. Firstly, we will prove a 4-poor vertex incident with f1, f2 and f3. Without loss of generality, suppose that all of f1, f2 and f3 are incident with a 4-poor vertex. Here, obviously we will assume that each of x, y and z is incident with a 4-poor vertex. We will consider a vertex by contraction of x, y and z. So, let N(a) = {a1, a2}. Continuously, we may construct each triangle incident with a such as u1x1x, u2yy1 and u3zz1. Then a2 is incident with both 5-face and 6-face. Let G 0 = {u, u1, u2, u3, a, a1, a2}. By minimality of G, suppose that G − G 0 has an (L, 1)∗−coloring of β. We will consider two cases. Case(i). We may assume that β(u1), β(u2) and β(u3) are the same colors and β(x), β(y) and β(z) are the same. So, we may assign the colors β(u1), β(u2) and β(u3) with 1 and then the colors β(x), β(y) and β(z) with 2. Here, we must assign the color β(u) with L(u) \ {β(u1), β(u2), β(u3)} and we must assign the color β(a1) with 3. Evidently, 5-face is 3-coloring and 6-face is 2-coloring. So, we must assign the colors β(a2) with 1. Here, we will assign the color β(u) with 3. Here, we must have all colors β(x), β(y) and β(z) with 2. If we exchange the colors β(u) and β(u1), we must recolor β(u2) with L(u2) \ {β(u 0 2 )}, β(u3) with L(u3) \ {β(u 0 3 )} and β(x1) with L(x1) \ {β(x 0 1 )}. Since β(x2) = 1, it must be β(x 0 1 ) = 1. Now, we can have the color β(x1) with 2. It is contradiction. Moreover, since u2 and u3

are incident with 6-face and we have that 6-face is 2-coloring, they must be the colors β(u 0 2 ) and β(u 0 3 ) with 2. So, we must have the colors β(u2) and β(u3) with 3. It is contradiction.

Furthermore, since |L(u)| = 3, we must assign the color β(u) with 2. If we exchange the colors β(u) and β(u1), we must recolor β(u2) with L(u2) \ {β(u 0 2 )} and β(u3) with L(u3) \ {β(u 0 3 )}. So, we must have the colors β(u2) and β(u3) with 3. Then, we will exchange the colors β(x) and β(x2). However, it is contradiction by assumption. Case (ii). We may assume that β(u1), β(u2) and β(u3) are different. Evidently, we must have the colors β(x), β(y) and β(z) are different. We may assume that the colors β(u1) with 1, β(u2) with 2 and β(u3) with 3. So, we must have the colors β(x) with 3, β(y) with 1 and β(z) with 2 and then continuously we must have the colors β(x1) with 2, β(y1) with 3 and β(z1) with 1. If we assign the color β(u) with 1, then we must recolor β(u1) with L(u1) \ {β(u 0 1 )}. Thus, we must have the color β(u1) with distinct β(u 0 1 ).

Here, it is contradiction. If we assign the color β(u) with 2, then we must recolor β(u2) with L(u2) \ {β(u 0 2 )}. Here, we must have the color β(u2) with distinct β(u 0 2 ). However, it is contradiction. If we assign the color β(u) with 3, then we must recolor β(u3) with L(u3)\ {β(u 0 3 )}. Here, we must have the color β(u3) with distinct β(u 0 3 ). However, it is contradiction. Finally, for the condition (ii) and (iii) are similar as the proof of the condition (i). Therefore, the proof is complete.

Corollary 2.12 Suppose to v is a 3-semi-poor vertex in which f1 = [vv1xv2], f2 = [vv2yv3] and f3 = [vv3zv1].

If the three vertices of x, y and z are 3- semi-poor vertices, then the three vertices of v1, v2 and v3 are 5 +−vertices. Lemma 2.13 Let v be 3-vertex, N(v) = {v1, v2, v3} and f = [vv1v2]. If v is a 3-full-poor vertex in which v1 and v3 are incident with 5-face, then (i) the three neighbors of v are 4 +−vertices (i.e., d(N(u)) ≥ 4) and (ii) exactly the vertex v1 is either a 4-poor vertex or a 5-poor vertex.

Definition 2.14 (i) A vertex u is a d(u)−vertex incident with at most n−triangles and others are any faces. Its vertex is called T nM−vertex. Here, |T nM| =the number of n−triangles incident with a vertex (ii) A vertex u is d(u)−vertex with d(u) ≥ 4 in which u is incident with exactly b d(u) 2 c 3-faces and exactly b d(u) 4 c 4-faces. It is said to be a T d(u)−vertex. Evidently, if d(u) is odd, then every 4-face must be incident between two 3-faces. Note that : If u is a 3-vertex incident with one 3-face and one 4-face or one 5-face, then another is one 8+−face. It is called T 1M−vertex. Lemma 2.15 Let u be T d(u)−vertex in G. Conditions

1. If u is T d(u)−vertex (d(u) = 3), then it is incident with distinct one 3-face, one 4-face and one 8 +−face. It is called a special T 3−vertex The following conditions: Let u be T d(u)−vertex in G with d(u) ≥ 4.
2. (ii) If u is T d(u)−vertex (d(u) = 4), then it is incident with distinct two 3-faces, one 4-face and one 8 +−face.
3. (iii) If u is T d(u)−vertex (where d(u) = 5), then it is incident with distinct two 3-faces, one 4-face, and then others are 5 +−faces. (iv) For d(u) ≥ 6, if u is a T d(u)−vertex and d(u) is odd, then it is incident with at most two 5 +−faces and others are incident with at most d d(u)−1 4 e − 1 8+−faces. (v) For d(u) ≥ 6, if u is a T d(u)−vertex and d(u) is even, then it is incident with at most d d(u) 4 e 8 +−faces.

 *T* 3 *T* 4 *T* 5

 *T* 6 T7 T8

Corollary 2.16 If u is a T d(u)−vertex (d(u) ≥ 7, d(u) = 4n+3, n = 1, 2, ...) in which there are incident with at most b d(u) 2 c 3-faces and at most b d(u) 4 c 4-faces, then there are at most two 5 +−faces and ( d(u) 4 − 3 4 ) 8+−faces.

Corollary 2.17 If u is a T d(u)−vertex (d(u) ≥ 9, d(u) = 4n+5, n = 1, 2, ...) in which there are incident with at most b d(u) 2 c 3-faces and at most b d(u) 4 c 4-faces, then there are at most two 5 +−faces and ( d(u) 4 − 5 4 ) 8+−faces.

**3 Discharging process**

We now apply a discharging procedure to reach a contradiction. We first define the initial charge function ch on the vertices and faces of G by letting ch(v) = ad(v) − 2b if v ∈ V (G) and ch(f) = (b − a)d(f) − 2b, f ∈ F(G). We note a = 3 2 and b = 7 2 so that we get the initial function ch(v) = 3 2 d(v)−7 if v ∈ V (G) and ch(f) = 2d(f) − 7, f ∈ F(G). It follows from Euler’s formula |V (G)| − |E(G)| + |F(G)| = 2 and the relation X v∈V (G) d(v) = X f∈F(G) d(f) = 2|E(G)|

 so that the total sum of initial function of the vertices and faces is equal to

X v∈V (G) ch(v) + X f∈F(G) ch(f) = X v∈V (G) ( 3 2 d(v) − 7) + X f∈F(G) (2d(f) − 7) = 3 2 (2|E(G)|) − 7|V (G)| + 2(2|E(G)|) − 7|F(G)| = −7(|V (G)| + |F(G)| − |E(G)|) = −14

Since any discharging procedure preserves the total charge of G, if we can define suitable discharging rules to change the initial charge function ch to the final charge function ch on V ∪F such that ch0 (x) ≥ 0 for all x ∈ V ∪F, then

0 ≤ X x∈V ∪F ch0 (x) = X x∈V ∪F ch(x) = −14,

a contradiction completing the proof of Theorem 1.1 when G is 2-connected.

Proof of Theorem 1.1 Since G is 2-connected, G has no adjacent 3-faces or 7-cycles and δ(G) ≥ 3, the following Lemma is obvious.

 Lemma 3.1 (i) In G, there is no adjacent 3-faces.

 (ii) In G, there is a 4-face adjacent to at most two 3-faces. Moreover, when a 4-face is adjacent to at least one 3-face, the 4-face can be adjacent to no 4-face except v is a 3-poor vertex.

1. In G, there is a 4-face adjacent to at least one 4-face.
2. (iv) In G, there is a 5-face adjacent to at most one 3-face and no adjacent to any 4-face. (v) In G, there is no 6-face adjacent to a 3-face. We will introduce the discharging rules: R 1. Charge from a 4+−face f R 1.1.
3. If d(f) = 4, then f sends 1 4 to each incident vertex. R 1.2. If d(f) = 5, then f sends 3 5 to each incident vertex.

R 1.3. If d(f) = 6, then f sends 5 6 to each incident vertex. R 1.4. If d(f) ≥ 8, then f sends 9 8 to each incident vertex.

R 2. Charge to a 3-face f = [v1v2v3] where d(v1) ≤ d(v2) ≤ d(v3). R 2.1. Suppose to v is a 4-light vertex. Let f = [v1v2v] = (5+, 3, 4)−face. Then v gets 9 8 from each 8 +−face and 1 4 from 4-face and it sends 3 2 to f. Then v2 gets 10 8 from 8+−face and 5 4 from f. After that v1 gets 9 8 from 8+−face and sends 13 16 to f.

R 3. Suppose to v is a poor vertex in which f = [v1v2v3] with d(v1) ≤ d(v2) ≤ d(v3).

R 3.1. Let d(v1) = 3 and v1 be a 3-poor vertex. Then v1 gets 1 2 from each 4-face and f sends 3 2 to v1.

R 3.2. Let d(v2) = 4 and v2 be a 4-poor vertex. v2 gets 3 5 from 5-face and 5 6 from 6-face and f gets 1 3 from v2.

R 3.3. Let d(v3) = 5 and v3 be a 5-poor vertex. v3 gets 3 5 from 5-face, 5 6 from 6+−face and 5 6 from 5+-face and then f gets 8 3 from v3.

R 4. Suppose to v be a 3-semi-poor vertex in which f1 = [vv1xv2], f2 = [vv2yv3] and f3 = [vv3zv1] with d(v) ≤ d(vi) where i ∈ {1, 2, 3}. R 4.1. Let d(v) = 3 and v be a 3-semi-poor vertex. Then v gets 5 6 from each 4-face.

 R 4.2. Let d(x) = d(y) = d(z) = 3 and they be 3-semi-poor vertices. So, vi a 5+-vertex where i ∈ {1, 2, 3}. Then v gets 1 2 from each 4-face and 1 3 from each 5+−vertex and 4-face sends 1 6 to other vertices not 3-semi-poor vertices. Moreover, x, y and z are like as v.

R 5. Suppose to v1 be a 3-full-poor vertex in which f = [v1v2v3] with d(v1) ≤ d(v2) ≤ d(v3). Then v1 gets 3 5 from 5-face and 18 8 from 8+-face and v1 sends 7 20 to f. Moreover, 8+-face sends 27 28 to other vertices.

 R 6. Suppose to v be a 4-semi-poor vertex in which f1 = [vv1v2], f3 = [vv3xv4] and f2 and f4 are 8+−faces with d(v1) = d(v4) = 3.

R 6.1 Let v be a 4-semi-poor I vertex. Then v gets 1 3 from f3 and 9 8 from 8+−face and it sends 3 2 to f1. R 6.1.1 For d(v1) = d(v4) = 3, v1 gets 9 8 from f1, 1 4 from 4-face and 9 8 from 8+−face and then v4 gets 2 3 from f3 and 9 8 from 8 +−face.

R 6.2 Let v be a 4-semi-poor II vertex. Then v gets 1 4 from f3 and 9 8 from 8+−face and it sends 3 2 to f1. R 6.2.1 For d(v1) = 3, v1 gets 9 8 from f1, 1 4 from 4-face and 9 8 from 8+−face

R 6.2.2 For d(v4) = 3, if the outer neighbor of v4 is 4-semipoor vertex, then v4 gets 3 4 from f3, 2 3 from 4-face and 9 8 from 8+−face. If the outer neighbor of v4 is not 4-semi-poor vertex, then v4 gets 3 4 from f3 and 1 4 from 4-face and 9 8 from 8 +−face.

R 6.3 Let v be a 4-semi-poor III vertex. Then v gets 1 3 from f3 and 9 8 from 8+−face and it sends 3 2 to f1.

R 6.3.1 For d(v1) = d(v4) = 3, v1 gets 7 8 from f1, 3 5 from 5-face and 9 8 from 8+−face and then v4 gets 2 3 from f3 and 9 8 from 8 +−face.

 R 6.4 Let v be a 4-semi-poor IV vertex. Then v gets 1 4 from f3 and 9 8 from 8+−face and it sends 3 2 to f1.

 R 6.4.1 For d(v1) = 3, v1 gets 7 8 from f1, 3 5 from 5-face and 9 8 from 8+−face.

R 6.4.2 For d(v4) = 3, if the outer neighbor of v4 is 4-semipoor vertex, then v4 gets 3 4 from f3, 2 3 from 4-face and 9 8 from 8+−face. If the outer neighbor of v4 is not 4-semi-poor vertex, then v4 gets 2 3 from f3 and 1 4 from 4-face and 9 8 from 8 +−face.

 R 7. Suppose to v be a 4-full-poor vertex in which f1 = [vv1xv2], f3 = [vv3yv4] and f2 and f4 are 8+−faces with d(v1) = d(v4) = 3. R 7.1 Let v be a 4-full-poor I vertex. Then v gets 9 8 from each 8 +−face and it sends 2 5 to v1 and v4.

R 7.1.1 For d(v1) = d(v4) = 3, both v1 and v4 get 1 2 from 4-face and 9 8 from 8+−face and then they get 2 5 from v. Moreover, f1 and f2 send 1 2 to 3-vertex and 1 8 to 4+−vertex. R 7.2 Let v be a 4-full-poor II vertex and v1 is incident with 3-face and v4 is incident with 4-face. Then v gets 9 8 from each 8+−face and it sends 2 5 to v1 and 1 5 to v4.

R 7.2.1 For d(v1) = 3, v1 gets 1 2 from 4-face and 9 8 from 8+−face and then it gets 2 5 from v. R 7.2.2 For d(v4) = 3, if the outer neighbor of v4 is 4-semi-poor vertex, then v4 gets 1 2 from f3 2 3 from 4-face and 9 8 from 8 +−face and then gets 1 5 from v. If the outer neighbor of v4 is not 4-semi-poor vertex, then v4 gets 1 2 from f3 and 1 4 from 4-face and 9 8 from 8+−face and then 1 5 from v. R 7.3 Let v be a 4-full-poor III vertex. Then v gets 9 8 from each 8 +−face and it sends 2 5 to both v1 and v4. R 7.3.1 For d(v1) = d(v4) = 3, if the outer neighbors of v1 and v4 is 4-semi-poor vertices, then both of v1 and v4 get 1 from each 4-face and 9 8 from 8+−face and then get 2 5 from v. If the outer neighbors of v1 and v4 are not 4-semi-poor vertices, then v1 and v4 get 1 from f1 and f3 and 1 4 from 4-face and 9 8 from 8+−face and then 2 5 from v.

R 8. Suppose to v is T d(v)−vertex. We deduce induction for d(v) ≥ 3. R 8.1. T 3 − vertex. Let f = [vv1v2] and v be 3-vertex incident with 4-face and 8+−face. If v is a T 3 vertex, then v gets 9 8 from 8+−face and 1 4 from 4-face. Then f sends 9 8 to v.

R 8.2. T 4 − vertex. If v is T 4 -vertex incident with one 4-face and one 8+−face, then v gets 10 8 from 8+−face and 1 4 from 4-face and then v sends 2 8 to each 3-face. R 8.3. T 5 − vertex Let f1 = [vv1v2] and f3 = [vv3v4]. v gets 3 5 from each 5+−face and 1 4 from 4-face. Then v sends 7 8 to each 3-face.

 R 8.4. T d(v) − vertex R 8.4.1 Let v be a T d(v)−vertex such that n is even and n ≥ 6. v gets 3 8 from each 8+−face and 1 4 from 4-face. In general v sends 53d(v)−224 16d(v) to each 3-face.

R 8.4.2 Let v be a T d(v)−vertex such that d(v) is odd and d(v) ≥ 7. Here v is incident with ( d(v) 4 − 3 4 ) 8+−face where d(v) = 4r + 3, r = 1, 2, . . . , n and d(v) ≥ 7 and incident with b d(v) 2 c 3-face and two 5+−face. Then v gets 3 8 from each 8+−face, 1 4 from 4-face and 3 5 from each 5+−face. In general for d(v) = 4n + 3, n = 1, 2, ..., and d(v) ≥ 7, v sends 52d(v)−194 16d(v) to each 3-face. (R 8.4.3) Let v be a T d(v)−vertex such that d(v) is odd and d(v) ≥ 9. Here v is incident with ( d(v) 4 − 5 4 ) 8+−face where d(v) = 4n+ 5, n = 1, 2, . . . , n and d(v) ≥ 9 and incident with b d(v) 2 c 3-face and two 5+−face. Then v gets 3 8 from each 8+−face, 1 4 from each 4-face and 3 5 from each 5+−face. In general for d(v) = 4n + 5, n = 1, 2, ..., and d(v) ≥ 9, v sends ( 52d(v)−202 16d(v) ) to 3-face.

R 9. For d(v) ≥ 4, if v is incident with 3-face, 4-face, 6+−face and 8+−face, then v gets 1 4 from 4-face, 5 6 from 6+−face and 9 8 from 8+−face and sends 1 to 3-face.

R 10. Otherwise, if v is not a poor vertex in which f = [v1, v2, v3] = (3, 4, 5)−face, then f gets 1 from 4-vertex and 3 2 from 5−vertex and then it sends 9 8 to v1. It remains to show that the resulting final charge ch0 is satisfied with ch0 ≥ 0 for all x ∈ V ∪F. Let v ∈ V (G) and f ∈ F(G). The proof can be completed-

with d(x) for all x ∈ V ∪ F. Let v ∈ V (G) and f ∈ F(G). Since d(v) ≥ 3. If d(v) = 4, by R 1 and R 2, then v is a 4-light vertex with f = (3, 4, 5 +)−face. So, ch0 (v) = ch(v) + 2 × 9 8 + 1 4 − 3 2 = 3 2 × 4 − 7 + 2 × 9 8 + 1 4 − 3 2 = 0 by R 2.1. Continuously, if d(v) = 3 by R 2.1 and R 5, then f = (3, 4, 5 +)−face and the 3-vertex is 3-full-poor vertex. So, ch0 (v) = ch(v) + 10 8 + 5 4 = 0 by R 2.1 and ch0 (v) = ch(v) + 10 8 + 5 4 + 3 5 > 0 R 5. If f = [v1v2v3] = (3, 4, 5) by R 1 and R 3 and by Lemma 2.8, then v1, v2 and v3 are 3-poor, 4-poor and 5-poor vertices. So, for d(v) = 3, ch0 (v) = ch(v) + 2 × 1 2 + 3 2 = 0 by R 3.1. And then for d(v) = 4, ch0 (v) = ch(v) + 3 5 + 5 6 − 1 3 = −1 + 3 5 + 5 6 − 1 3 ≥ 0 R 3.2. Moreover, for d(v) = 5, ch0 (v) = ch(v)+ 3 5 + 2× 5 6 − 8 3 = 1 2 + 3 5 + 2× 5 6 − 8 3 ≥ 0 R 3.3. If d(v) = 3 and f1 = [vv1xv2], f2 = [vv2yv3] and f3 = [vv3zv1], then v is a 3-semi-poor vertex by R 1 and R 4 . So, we have ch0 (v) = ch(v) + 3× 5 6 = 3 2 ×3−7 + 3× 5 6 = 0 by R 4.1. By Corollary 2.12 if d(x) = d(y) = d(z) = 3 and they are 3-semipoor vertices, then d(vi) ≥ 5. So, ch0 (v) = ch(v)+3× 1 2 +3× 1 3 = − 5 2 + 5 2 = 0 by R 4.2. If d(v) = 3 and f = [vv1v2] = (3, 4, 4 +) and N(v) = {v1, v2, v3} by R 1 and R 5 and by Lemma 2.13, then v is a 3-full-poor vertex. So, ch0 (v) = ch(v) + 3 5 + 18 8 − 7 20 = − 5 2 + 5 2 = 0 by R 5. Then, if v1 is a 4-poor vertex, then v2 is incident with 4-face, 6+−face and 8+−face. So, for d(v) ≥ 4, ch0 (v) = ch(v) + 1 4 + 5 6 + 27 28 − 1 ≥ 0 by R 9 and R 5. Here, for 3-face, ch0 (f) = ch(f) + 1 3 + 7 20 + 1 > 0 R 3.2 and R 5 and R 9. For d(v) = 4, if f1 = [vv1v2], f3 = [vv3xv4] and f2 and f4 are 8+−faces with d(v1) = d(v4) = 3, then v is a 4-semi-poor vertex by R 1 and R 6. If v is a 4-semi-poor vertex I, then ch0 (v) = ch(v) + 1 3 + 2 × 9 8 − 3 2 = −1 + 1 3 + 9 4 − 3 2 > 0 by R 6.1. For d(v1) = 3, we must have d(v2) ≥ 4. So, ch0 (v1) = ch(v1) + 9 8 + 1 4 + 9 8 = 0 by R 6.1.1 and R 9. Then f = [vv1v2], ch0 (f) = ch(f)+ 3 2 + 1− 9 8 > 0 by R 6.1, R 6.1.1 and R 9. For d(v4) = 3, if v4 is incident with f = (3, 4, 5)−face, then ch0 (v4) = ch(v4) + 2 3 + 9 8 + 9 8 > 0 by R 6.1.1 and R 10. If v is a 4-semi-poor vertex II, then ch0 (v) = ch(v) + 1 4 + 2 × 9 8 − 3 2 = −1 + 1 3 + 9 4 − 3 2 = 0 by R 6.2. For d(v4) = 3, if the outer neighbor of v4 is 4-semi-poor vertex, then ch0 (v4) = ch(v4) + 2 3 + 3 4 + 9 8 ≥ 0 by R 6.2.2. For d(v4) = 3, if the outer neighbor of v4 is 4-full-poor vertex, then ch0 (v4) = ch(v4) + 1 4 + 3 4 + 2 5 + 9 8 ≥ 0 by R 6.2.2 and R 7.1 If v is a 4-semi-poor vertex III, then ch0 (v) = ch(v) + 1 3 + 2 × 9 8 − 3 2 = −1 + 1 3 + 9 4 − 3 2 > 0 by R 6.3. For d(v1) = 3, we must have d(v2) ≥ 4. So, ch0 (v1) = ch(v1) + 7 8 + 3 5 + 9 8 > 0 by R 6.3.1 and R 9. Then f = [vv1v2], ch0 (f) = ch(f) + 3 2 + 1− 7 8 > 0 by R 6.3, R 6.3.1 and R 10. For d(v4) = 3, if v4 is incident with f = (3, 4, 5)−face, then ch0 (v4) = ch(v4)+ 2 3 + 9 8 + 9 8 > 0 by R 6.3.1 and R 10. If v is a 4-semi-poor vertex IV, then ch0 (v) = ch(v) + 1 4 + 2 × 9 8 − 3 2 = −1 + 1 3 + 9 4 − 3 2 = 0 by R 6.4. For d(v4) = 3, if the outer neighbor of v4 is 4-semi-poor vertex, then ch0 (v4) = ch(v4) + 2 3 + 3 4 + 9 8 ≥ 0 by R 6.4.2. For d(v4) = 3, if the outer neighbor of v4 is 4-full-poor vertex, then ch0 (v4) = ch(v4) + 1 4 + 3 4 + 2 5 + 9 8 ≥ 0 by R 6.4.2 and R 7.1 For d(v) = 4, if f1 = [vv1xv2], f3 = [vv3yv4] and f2 and f4 are 8+−faces with d(v1) = d(v4) = 3, then v is a 4-full-poor vertex by R 1 and R 7. If v is a 4-full-poor vertex I, then ch0 (v) = ch(v) + 1 8 + 2 × 9 8 − 2 × 2 5 =

−1 + 9 4 − 4 5 > 0 by R 7.1. For d(v1) = d(v4) = 3, if v1 and v4 are incident with f = (3, 4, 5), then ch0 (v) = ch(v) + 1 2 + 2 5 + 9 8 + 9 8 > 0 by R 7.1.1 and R 11 (where v is represented by v1 and v4). If v is a 4-full-poor vertex II, then ch0 (v) = ch(v) + 1 8 + 2 × 9 8 − 2 5 − 1 5 = −1 + 1 8 + 9 4 − 3 5 > 0 by R 7.2. For d(v4) = 3, if the outer neighbor of v4 is 4-semi-poor vertex, then ch0 (v4) = ch(v4) + 1 2 + 1 5 + 9 8 + 2 3 = 0 by R 7.2.2 and R 6.1. For d(v4) = 3, if the outer neighbor of v4 is 4-full-poor vertex, then ch0 (v4) = ch(v4) + 1 4 + 1 2 + 1 5 + 2 5 + 9 8 = 0 by R 7.2.2 and R 7.1

For d(v) = 3, by R 1 and R 8, if v is incident with 3-face, 4-face and 8+−face, then v is a T 3−vertex. Let f = [vv1v2] = (3, 4, 4 +)−face. Here, v is T 3−vertex and we can get v1 is a 4-semi-poor vertex and v2 ≥ 4 and so ch0 (v) = ch(v) + 1 4 + 9 8 + 9 8 = 0 by R 8.1, R 6 and R 9. Then ch0 (f) = ch(f) + 3 2 + 1 − 9 8 > 0 by R 8.1, R 6 and R 9. For d(v) = 4, by R 1 and R 8, if v is incident with two 3-faces, one 4-face and one 8+−face, then v is a T 4−vertex. Let f1 = [vv1v2] and f3 = [vv3v4], f2 be 4-face and f4 is 8+−face. So, ch0 (v) = ch(v)+1 4+ 10 8 −2× 2 8 = 0 by R 8.2. Let f1 = f3 = (3, 4, 5). If v is a T 4−vertex, then ch0 (f) = ch(f) + 2 8 + 3 2 − 9 8 < 0 by R 8.2, R 10 or ch0 (f) = ch(f) + 2 8 + 3 2 − 3 2 < 0 by R 8.2, R 3.1. So, it is impossible that T 4−vertex is adjacent to 3-vertex.

Lemma 3.2 Let f1 = [vv1v2] and f3 = [vv3v4], f2 be 4-face and f4 is 8 +−face. If v is a T 4−vertex, then the neighbor vertices of v are 4 +−vertex.

For d(v) = 5, by R 1 and R 8, if v is incident with two 3-faces, one 4-face and two 5+−face, then v is a T 5−vertex. Let f1 = [vv1v2 and f3 = [vv3v4], f2 be 4-face and f4 and f5 are 5+−faces. So, ch0 (v) = ch(v)+1 4+2× 3 5−2× 7 8 = 0 by R 8.3. For f1 and f2, if v is a T 5−vertex, then ch0 (f) = ch(f)+7 8+1− 3 2 < 0 by R 8.3, R 9 and R 3.1 or ch0 (f) = ch(f) + 7 8 + 3 2 − 3 2 < 0 by R 8.2, R 3.1 and R 10. So, it is impossible that T 5−vertex is adjacent to 3- poor vertex. Then ch0 (f) = ch(f) + 7 8 + 3 2 − 9 8 > 0 by R 8.2, R 10 and ch0 (f) = ch(f) + 7 8 + 1 − 9 8 < 0 by R 8.2, R 9. Therefore, if v is T 5−vertex adjacent to T 3−vertex, then f = (5, 3, 5 +)−face.

Lemma 3.3 In G, let v be a T 5−vertex in which f1 = [vv1v2 and f3 = [vv3v4], f2 be 4-face and f5 be 5 +−faces. If a T 5−vertex is adjacent to T 3−vertex, then f1 = f2 = (5, 3, 5 +)−face.

Moreover, if v is a T d(v)−vertex, where d(v) ≥ 6 and d(v) is even, by Lemma 2.15, then v is incident at most b d(v) 2 c 3-faces, at most b d(v) 4 c 4-faces and at most d d(v) 4 e 8 +−faces. So, by R 1 and R 8,

ch0 (v) ≥ ch(v) + 3 8 (d d(v) 4 e) + 1 4 (b d(v) 4 c) − 53d(v) − 224 16d(v) b d(v) 2 c = 3 2 d(v) − 7 + d 3d(v) 32 e + b 2d(v) 32 c − 53d(v) − 224 16d(v) b d(v) 2 c = 53d(v) − 224 32 − 53d(v) − 224 16d(v) b d(v) 2 c ≥ 0

by R 8.4.1. If v is a T d(v)−vertex (d(v) ≥ 7, d(v) = 4n + 3, where n = 1, 2, ...) by R 8.4.2 and by Corollary 2.16, then ch0 (v) ≥ ch(v) + 3 8 ( d(v) 4 − 3 4 ) + 1 4 (b d(v) 4 c) + 2 × 3 5 − ( 52d(v) − 194 16d(v) )b d(v) 2 c = 3 2 d(v) − 7 + 3d(v) 32 + b d(v) 16 c + 6 5 − 9 32 − 52d(v) − 194 16d(v) b d(v) 2 c = 51d(v) 32 + b d(v) 16 c − 973 160 − 52d(v) − 194 16d(v) b d(v) 2 c ≤ 265d(v) − 973 160 − 52d(v) − 194 32 = 265d(v) − 973 160 − 260d(v) − 970 160

> 0 If v is a T d(v)−vertex (d(v) ≥ 9, d(v) = 4n + 5, where n = 1, 2, ...) by R 8.4.3 and by Corollary 2.17, then ch0 (v) ≥ ch(v) + 3 8 ( d(v) 4 − 5 4 ) + 1 4 (b d(v) 4 c) + 2 × 3 5 − ( 52d(v) − 202 16d(v) )b d(v) 2 c = 3 2 d(v) − 7 + 3d(v) 32 + b d(v) 16 c + 6 5 − 15 32 − 52d(v) − 202 16d(v) b d(v) 2 c = 51d(v) 32 + b d(v) 16 c − 1018 160 − 52d(v) − 202 16d(v) b d(v) 2 c ≤ 265d(v) − 1018 160 − 52d(v) − 202 32 = 265d(v) − 1018 160 − 260d(v) − 1010 160 > 0 If v is a 4-light vertex, then f = [v1v2v] = (3, 3, 4)−face by R1 and R2.1 and R 5. If v1 and v2 are 3-full-poor vertices, then ch0 (f) = ch(f) + 1 + 3 4 + 7 20 = 2d(f) − 7 + 22 20 ≥ 0. By Lemma 2.9, when d(f) = 4, f sends 1 4 to each 4-light vertex. ch0 (f) = ch(f) − 4 × 1 4 = 0 by R 2.1 and R 1. Suppose d(f) = 3 with f = [v1v2v3] = (3, 4, 5)−face. By Lemma 2.8 and R 3, if v1, v2 and v3 are poor vertices, then ch0 (f) = ch(f)+ 1 3 + 8 3 − 3 2 = 2d(f)−7+ 3 2 > 0 by R 3.1, R 3.2 and R 3.3. By R 10, if v1, v2 and v3 are not poor vertices

then ch0 (f) = ch(f) + 3 2 + 1 − 9 8 = 2d(f) − 7 + 11 8 > 0. For d(f) = 4, by Lemma 2.11, ch0 (f) = ch(f) − 5 6 − 1 3 = 2d(f) − 7 − 7 6 < 0 by R 3.2, R 4.1 and R 6.1. So, Lemma 2.11 is true. We have that G is simple, has neither adjacent triangles nor 7-cycles and δ(G) ≥ 3, the following lemma is obvious. This completes the proof of Theorem 1.1.

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