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EVERY PLANAR GRAPH WITHOUT ADJACENT TRIANGLES OR 7-CYCLES IS $(3,1)^*$ - CHOOSABLE

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ABSTRACT

In a graph G, a list assignment L is a function that it assigns a list $L(v)$ of colors to each vertex $v \in V(G)$. An $(L,d)^*$ −coloring is a mapping β that assigns a color $\beta(v) \in L(v)$ to each vertex $v \in V(G)$ so that at most impropriety d neighbors of v are the same color with $\beta(v)$. A graph G is said to be $(k,d)^*$ − choosable if it admits an $(L,d)^*$ −coloring for every list assignment L with $|L(v)| \geq k$ for all $v \in V(G)$. In this paper, we prove that every planar graph with neither adjacent triangles nor 7-cycles is $(3,1)^*$ −choosable. In 2016, Min Chen, Andre Raspaud and Weifan Wang proved that every planar graph with neither adjacent triangles nor 6-cycles is $(3,1)^*$ −choosable.

Keywords: Planar Graphs, Improper Choosability, Cycle.

1. INTRODUCTION

A $k-$ coloring of G is a mapping β from $V(G)$ to a color set $\{1,2,...,k\}$ such that $\beta(x) \neq \beta(y)$ for any adjacent vertices x and y. A graph is $k-colorabe$ if it has a $k-coloring$. Cowen et al.(1986) considered defective coloring of graphs. A graph G is said to be $d-improper$ $k-colorable$, or simply, $(k,d)^*-colorable$, if the vertices of G can be colored with k colors in such a way that vertex has at most d neighbors receiving the same color as itself. Clearly, a $(k,0)^*-coloring$ is an ordinary proper $k-coloring$.

A list assignment of G is a function L that assigns a list $L(v)$ of colors to each vertex $v \in V(G)$. An $L-coloring$ with impropriety of integer d, or simply an $(L,d)^*-coloring$, of G is a mapping β that assigns a color $\beta(v) \in L(v)$ to each vertex $v \in V(G)$ so that at most d neighbors of v receive color $\beta(v)$. A graph is $k-choosable$ with impropriety of integer d, or simply $(k,d)^*-choosable$, if there exists an $(L,d)^*-coloring$ for every is just the ordinary $k-$choosability introduced by Erdös et al. (1979) and independently by Vizing (1976). A famous and classic result given by Thomassen (1994) is that every planar graph is $(5,0)^*$ −choosable. However, Voigt (1993) showed that not all planar graphs are $(4,0)^*$ −choosable by establishing a non $-(4,0)^*$ −choosable planar graph.

In 1999, Šrekovski(1999a) and Eaton and Hull (1999) independently introduced the concept of list improper coloring. They showed that planar graphs are $(3,2)^*$ −choosable and outerplanar graphs are $(2,2)^*$ −choosable. They are both improvement of the results shown in Cowen et al. (1986) which say that planar graphs are $(3,2)^*$ − colorable and outerplanar graphs are $(2,2)^*$ colorable. Note that there exist non $-(2,2)^*$ −colorable planar graphs and non $-(2,1)^*$ −colorable outerplanar graphs which were constructed in Cowen et al. (1986). Let $g(G)$ denote the girth of a graph G, i.e., the length of a shortest cycle in G. The $(k,d)^*$ −choosability of planar graph G with given $g(G)$ has been investigated by Šrekovski (2000). He proved that every planar graph G is $(2,1)^*$ − choosable if $g(G) \geq 9$, $(2,2)^*$ − choosable if $g(G) \geq 7$, $(2,3)^*$ −choosable if $g(G) \geq 6$, and $(2,d)^*$ − choosable if

$d \geq 4$ and $g(G) \geq 5$. The first two results were strengthened by Havet and Sereni (2006) who proved that every planar graph G is $(2,1)^*-$ choosable if $g(G) \geq 8$ and $(2,2)^*-$ choosable if $g(G) \geq 6$. Recently, Cushing and Kierstead (2010) proved that every planar graph is $(4,1)^*-$ choosable. So it would be interesting to investigate the sufficient conditions of $(3,1)^*-$ choosability of subfamilies of planar graphs where some families of cycles are forbidden. Šrekovski proved in Šrekovski (1999b) that every planar graph without 3-cycles is $(3,1)^*-$ choosable. Lih et al. (2001) proved that planar graphs without 4- and $l-$cycles are $(3,1)^*-$choosable, where $l \in \{5,6,7\}$. Later, Dong and Xu (2009) proved that planar graphs without 4- and $l-$ cycles are $(3,1)^*-$ choosable, where $l \in \{8,9\}$. These two results were improved further by Wang and Xu (2013) who showed that every planar graph without 4-cycles is $(3,1)^*-$ choosable. More recently, Chen and Raspaud (2014) proved that every planar with neither adjacent 4-cycles nor 4-cycles adjacent to 3-cycles is $(3,1)^*-$ choosable. This absorbs above results in Lih et al. (2001), Dong and Xu (2009), Wang and Xu (2013). Then, Min Chen, Andre Raspaud and Weifan Wang (2016) proved that every planar graph with neither adjacent triangles nor 6-cycles is $(3,1)^*-$ choosable.

Theorem 1.1 Every planar graph with neither adjacent triangles nor 7-cycles is $(3,1)^*-$ choosable.

The proof of Theorem 1.1 is done in the section 3.

2. NOTATION

All graphs considered in this paper are finite, simple and undirected without multiple edges. Call a graph G planar if it can be embedded into the plane so that its edges meet only at their ends. Any such particular embedding of a planar graph is called a plane graph. For a plane graph G, we use V, E, F, Δ and δ $(V(G), E(G), F(G), \Delta(G), \delta(G))$ to denote its vertex set, edge set, face set, maximum degree and minimum degree, respectively. For a vertex $v \in V$, the degree of v in G, denoted by $d_G(v)$, or simply $d(v)$, is the number of edges incident with v in G. $|V(G)|$ and $|E(G)|$ are order and size. The neighborhood of v in G, denoted by $N_G(v)$, or simply $N(v)$, consists of all vertices adjacent to v in G. Call v a $k-$vertex, or a k^+- vertex, or a $k^- -$ vertex if $d(v) = k$, or $d(v) \geq k$, or $d(v) \leq k$, respectively. A similar notation will be used for cycles and faces. For a face $f \in F$, the number of edges of the boundary of f (where cut edge, if any, is counted twice), denoted by $d(f)$, is called the degree of f. Analogously, the notations above for vertices will be applied to faces. We write $f = [v_1 v_2 ... v_k]$ if $v_1, v_2, ..., v_k$ are consecutive vertices on f in a cyclic order, and say that f is a $(d(v_1), d(v_2), ..., d(v_k))-$face. Next, let f_i be the face with vv_i and vv_{i+1} as two boundary edges for $i = 1, 2, ..., d(v)$, where indices are taken modulo $d(v)$ and define $d(v)+1 = 1$. Let v be a vertex, and v is a 3-vertex in G such that the three neighbors vertices adjacent with v. An edge xy is called a $(d(x), d(y))-$edge, and x is called a $d(x)-$neighbor of y. A $k-$cycle is a cycle of length k. In this paper, a 3-face is often called a triangle. Call a vertex or an edge triangular if it is incident with a triangle. Otherwise, a vertex or an edge **iso-triangular** if it is not incident with a triangle but its neighbor vertex is incident with triangle. Then 4-face is often called a quadrilateral. Two cycles or two faces are intersecting if they have at least one vertex in common; and are adjacent if they have at least one edge in common. Again, 4-face is called a quadrilateral in which two triangles are adjacent. We define the following notation:

- Let u be a 4-vertex. If u is incident with f_1, f_2, f_3 and f_4 so that $f_1 = [uu_1u_2] = (3,4,5^+)-$face and then $d(f_3) = 4$ and $d(f_2) = d(f_4) = 8^+ -$ face. It is called **4-light vertex**. Shown in Figure 1.

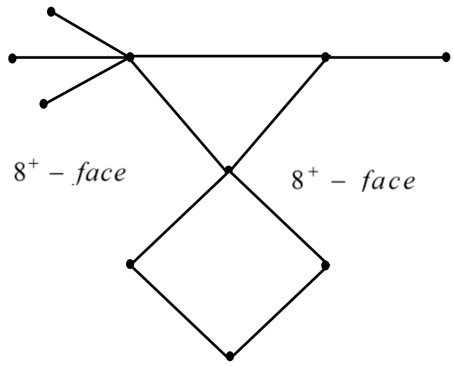

Figure 1:

Definition 2.1 Let f be 3-face such that $f = [uu_1u_2]$ and e_f be an edge incident with f.

i.e., e_{uu_1}, e_{uu_2}, $e_{u_1u_2}$ can be written by e_f.

Definition 2.2

- A 3-vertex is said to be **poor** if it is incident with one 3-face and two 4-faces. Then it is called **3-poor.**
- Let u be a 4-vertex and $f = [uu_1u_2]$ be a 3-face. If u is incident with one 3-face, one 4-face and one 5 face adjacent with e_f and another is 6-face, then it is said to be 4-**poor**. (OR)
- A 4-vertex is said to be **poor** if it is incident with one 3-face and two of e_f incident with one 4-face and one 5-face and another is 6-face. Then it is called **4-poor**.
- Let u be a 5-vertex and $f = [uu_1u_2]$ be a 3-face. If u is incident with one 3-face and both one 4-face and one 5-face adjacent with e_f and others' two are $6^+ -$face and $5^+ -$face, then it is said to be **5-poor**.

(OR)

A 5-vertex is said to be **poor** if it is incident with one 3-face and two of e_f incident with one 4-face and one 5-face and others are incident with $6^+ -$ face and $5^+ -$face. Then it is called **5-poor**.

Definition 2.3

- A 3-vertex is said to be **semi-poor** if it is incident with three 4-faces. Then it is called **3-semi-poor**.
- A 4-vertex is said to be **semi-poor** if it is incident with one 3-face adjacent to one 4-face and one 4-face adjacent to one 3-face. Then it is also called a **semi-poor-I** vertex.
- A 4-vertex is said to be **semi-poor** if it is incident with one 3-face adjacent to one 4-face and one 4-face adjacent to one 4-face. Then it is also called a **semi-poor-II** vertex.
- A 4-vertex is said to be **semi-poor** if it is incident with one 3-face adjacent to one 5-face and one 4-face adjacent to one 3-face. Then it is also called a **semi-poor-III** vertex.
- A 4-vertex is said to be **semi-poor** if it is incident with one 3-face adjacent to one 5-face and one 4-face adjacent to one 4-face. Then it is also called a **semi-poor-IV** vertex.

Definition 2.4

- A 3-vertex is said to be **full-poor** if it is incident with one 3-face, one 5-face and $8^+ -$face. Then it is called **3-full-poor**.
- A 4-vertex is said to be **full-poor** if it is incident with one 4-face adjacent to one 3-face and one 4-face adjacent to one 3-face. Then it is also called a **full-poor-I** vertex.
- A 4-vertex is said to be **full-poor** if it is incident with one 4-face adjacent to one 3-face and one 4-face adjacent to one 4-face. Then it is also called a **full-poor-II** vertex.

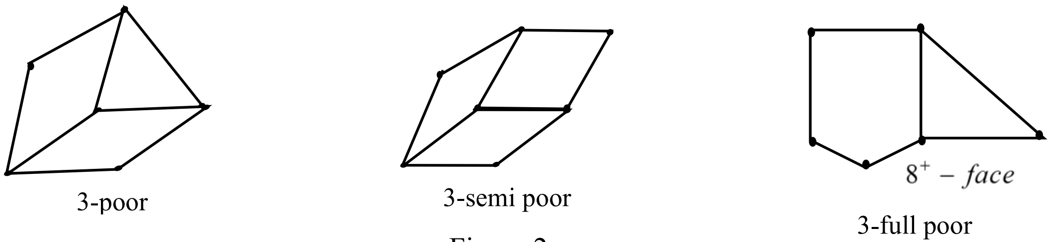

Figure 2:

- A 4-vertex is said to be **full-poor** if it is incident with one 4-face adjacent to one 4-face and one 4-face adjacent to one 4-face. Then it is also called a **full-poor-III** vertex.

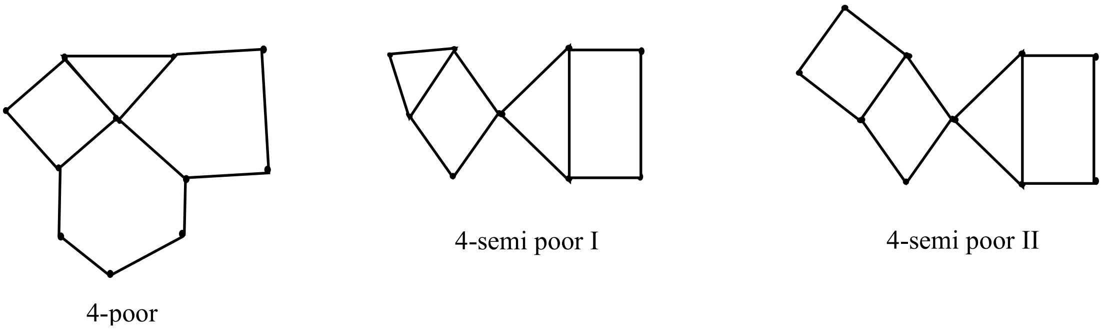

Figure 3:

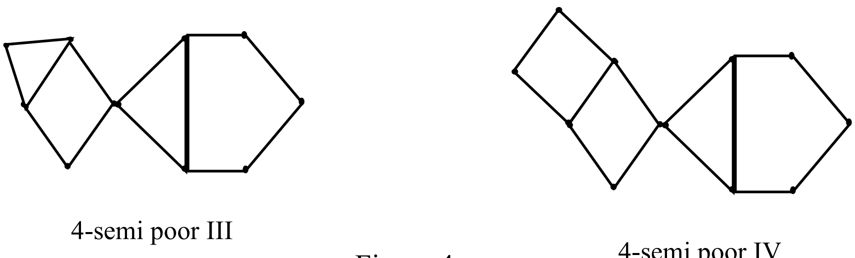

Figure 4:

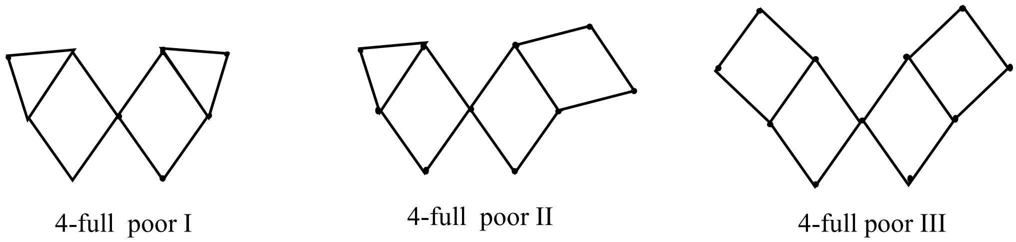

Figure 5:

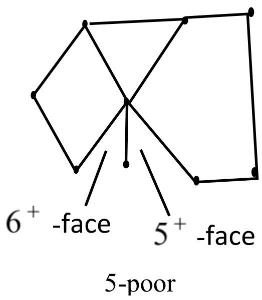

Figure 6:

Theorem 2.5 (Chen [1]). Every planar graph neither adjacent triangle nor 6 cycle is $(3,1)^* -$choosable.

Theorem 2.6 (Chen [2]). Every planar graph without 4-cycles adjacent to 3- and 4-cycles is $(3,1)^* -$choosable.

Lemma 2.7 (Lih, Wang, Zhang [9]).

(A 1) $\delta(G) \geq 3$.

(A 2) No two adjacent 3-vertices.

Lemma 2.8 Let f be (3,4,5)-face. Then all vertices of f are poor.

Proof: Let $f = [xyz] = (3,4,5) -$ face and then $x_1 \in N(x)$, $y_1, y_2 \in N(y)$ and $z_1, z_2, z_3 \in N(z)$. Suppose to the contrary that there is no poor vertex of f in G. Let $G' = \{x, y, z, x_1, y_1, y_2, z_1, z_2, z_3\}$. By minimality of G, suppose that $G - G'$ has an $(L,1)^* -$coloring of β.

First, for $d(x) = 3$, without loss of generality, let xx_1y_1y be a quadrilateral and e_{xz} be not incident with 4-face. We may provide the colors $\beta(y) = \beta(x_1) = \beta(z_1) = 1$ and $\beta(y_1) = \beta(z) = 2$. We must have the color $\beta(x)$ with $L(x) \quad \{\beta(y) \bigcup \beta(z) \bigcup \beta(x_1)\}$. So, we choose the color $\beta(x)$ with 3. If we recolor $\beta(x_1)$ with $L(x_1) \quad \{\beta(y_1) \bigcup \beta(x_1')\}$, then we will get the color of the same $\beta(x)$. If we recolor $\beta(x_1)$ with 3, we can exchange the colors $\beta(x)$ and $\beta(z)$ However, since e_{xz} is not incident with 4-face, it means that it is incident with 8-face. So, y_1 and x_1' can be adjacent to each other. If $y_1x_1x_1'$ is a triangle, we must have the color $\beta(x_1')$ with 3. So, it is impossible for the color $\beta(x_1)$ with 3. If $y_1x_1x_1'$ is not a triangle, y_1yy_2 can be a triangle. So, we can assume that the colors $\beta(x_1)$ and $\beta(y_2)$ with 3. Since e_{xz} is not incident with 4-face, so $x_1' \neq z_1$. So, we could have the colors $\beta(x_1')$ and $\beta(z_1)$ are the same. Then we change the colors $\beta(z)$ and $\beta(z_1)$. It is contradiction for x vertex.

Secondly, for $d(y) = 4$ and $d(z) = 5$, we have proved that x is a poor vertex. Without loss of generality, we have x_1xyy_1 and x_1xzz_1 are quadrilaterals and then we cannot have both yy_1y_2 is a triangle and $yy_1 * y_2$ is a quadrilateral. So, we may assume that zz_2z_3 is a triangle. Since e_{yz} is not incident with 4-,5-,6-faces. Without loss of generality, let $L(x) = L(y_1) = L(y_2) = L(z_1) = \{1,2,3\}$, $L(y) = L(z_2) = \{1,2,4\}$, $L(z) = L(x_1) = \{1,3,4\}$ and $L(z_3) = \{2,3,4\}$. If we provide the colors $\beta(y_1) = \beta(y_2) = \beta(z_2) = 1$, $\beta(z_1) = 3$ and $\beta(y) = \beta(z_3) = 2$, then we must have the colors $\beta(x_1)$ with 4 and $\beta(z)$ with 4. We can give the color $\beta(x)$ with $L(x) \quad \{\beta(y) \bigcup \beta(z) \bigcup \beta(x_1)\}$. If we recolor $\beta(y)$ with 4, we must exchange the colors $\beta(z_3)$ and $\beta(z)$. However, $2 \notin L(z)$. It is impossible. Thus, it is contradiction by assumption. Therefore, the proof is complete.

Lemma 2.9 If f be a (4,4,4,4)-face, then every vertex of 4-face can be a 4-light vertex.

Proof: Let $[xyzw]$ be a 4-face in which every vertex is a 4-vertex. Assume that x_i, y_i, z_i and w_i are the neighbors of x, y, z, w, composing of a triangle with their neighbors where $i \in \{1,2\}$. Suppose to the contrary that none of x, y, z, w is a 4-light vertex such that $d(A_i) \geq 4$, where $A_i = \{x_i, y_i, z_i, w_i\}$, $i = \{1,2\}$. Let $G' = \{x, y, z, w, x_i, y_i, z_i, w_i\}$, $i = \{1,2\}$. By the minimality of G, $G - G'$ admits an $(L,1)^* -$ coloring of β.

We will consider two cases.

Case (i) We may give colors with $\beta(x)$ and $\beta(z)$ are the same and $\beta(y)$ and $\beta(w)$ are also. So, let $\beta(x)=\beta(z)=1$ and $\beta(y)=\beta(w)=2$. Thus, we can deduce that $\beta(a_i)\in\{2,3\}$ and $\beta(b_i)\in\{1,3\}$, where $a_i=\{x_i,z_i\}$ and $b_i=\{y_i,w_i\}$, $i\in\{1,2\}$. We consider three sub-cases in the following.

Sub-case (i) Firstly, for x we will consider x_1 and x_2 have to be incident with only one triangle. By assumption, we have $[x_1x_2x]=(3,4,4)-$face. We must have the colors $\{\beta(x_1'),\beta(x_2'),\beta(x_2'')\}\subseteq\{1,2,3\}$. If $x_1x_1'x_2'x_2$ is a quadrilateral, we cannot give the same colors $\beta(x_1')$, $\beta(x_2')$ and $\beta(x_2'')$. So, we may assume that $\beta(x)=\beta(x_2'')=1$, $\beta(x_1)=\beta(x_1')=2$, $\beta(x_1')=\beta(x_1'')=\beta(x_2'')=1$. Here, we must have the colors $\beta(x_2')=2$. If we exchange the colors $\beta(x_2)$ and $\beta(x_2'')$, we must recolor $\beta(x)$ with 2 or 3. Clearly, $\beta(x)=2$ is impossible. So, we must have the color $\beta(x)$ with 3. Moreover, secondly, for the vertex y we will consider y_1 and y_2 have to be incident with only one triangle. We may assume that $\beta(y_1)=1$, $\beta(y_2)=3$. If $y_1y_1'y_2'y_2$ is a quadrilateral, we have different colors between y_1' and y_2'. So, if we assume that $\beta(y_2')=\beta(y_2'')=2$, we must have the colors $\beta(y_1')$ with 3. Clearly, we have $\beta(y_1)=1$ and $\beta(y_2)=3$ If we exchange the colors $\beta(y_2)$ and $\beta(y_2'')$. We must recolor $\beta(y)$ with 3. It is contradiction by assumption.

Sub-case (ii) For the vertex x, we will consider x_1 and x_2 have to be incident with triangle. We must have the colors $\{\beta(x_1'),\beta(x_2')\beta(x_2'')\}\subseteq\{1,2,3\}$. Let $x_2x_2'x_2''$ be a triangle and be a $x_1x_1'x_2'x_2$ quadrilateral. We may assume that $\beta(x_1)=2$, $\beta(x_2)=3$, $\beta(x_1')=\beta(x_2'')=1$. Here, we must have the color $\beta(x_2')=2$. If we exchange the colors $\beta(x_1)$ and $\beta(x_1')$, and then the colors $\beta(x_2)$ and $\beta(x_2')$, we must recolor $\beta(x)$ with 3. Moreover, for the vertex y we will consider y_1 and y_2 have to be incident with triangle. Let $y_2y_2'y_2''$ be a triangle and $y_1y_1'y_2'y_2$ be a quadrilateral. We may assume that $\beta(y_1)=1$, $\beta(y_2)=3$ and $\beta(y_1')=\beta(y_2'')=2$. So, we must have the color $\beta(y_2')=1$. If we exchange the colors $\beta(y)$ and $\beta(y_1)$, it is impossible for $\beta(y_1')\subseteq\{1,3\}$. Thus, we will exchange the colors $\beta(y)$ and $\beta(y_2)$. It is contradiction by assumption.

Sub-case (iii) For the vertex x, we will consider x_1 and x_2 to be incident with three triangles. Obviously, x_1 and x_2 do not be incident with any quadrilateral. Let $\beta(x_1)=\beta(x_2')=2$ and $\beta(x_1')=3$. We must have the colors $\beta(x_2)$ with 3 and $\beta(x_2'')$ with 1. Similarly, we will consider the vertex y. Let $\beta(y_1)=\beta(y_2')=1$ and $\beta(y_1')=2$. We must obtain the colors $\beta(y_2)$ with 3 and $\beta(y_2'')$ with 2. If we recolor any vertex, it is very strict. Since x_i and y_i where $i\in\{1,2\}$, are incident with only 8^+-face, any neighbor of x_1', x_2' and x_2'' and any neighbor of y_1' and y_2'' cannot be adjacent to each other. Here, $(L,1)-$coloring is satisfied. Thus, it is contradiction. It is enough to prove only two vertices x and y.

Case(ii) We may give colors with $\beta(x)$ and $\beta(y)$ are different. So, let $\beta(x)=1$ and $\beta(z)=2$ and $\beta(y)=3$ and $\beta(w)=a$. We must have the colors $\beta(x_i)\in\{2,3\}$, $\beta(y_i)\in\{1,2\}$, and $\beta(z_i)\in\{1,3\}$, where $i\in\{1,2\}$. Suppose that $a=3$. We must have $\beta(w_i)\in\{1,2\}$. If we exchange the colors $\beta(x)$ and $\beta(x_1)$, we must have colors $\beta(x)\in\{2,3\}$. If we have the colors $\beta(x)$ with 3, it is impossible because of $\beta(y)=3$. So, there is the color $\beta(x)$ with 2. If we exchange the colors $\beta(y)$ and $\beta(y_1)$, we must have colors $\beta(y)\in\{1,2\}$. If we have a color $\beta(y)$ with 2, it is impossible. So, there must be the color $\beta(y)$ with 1. If we exchange the colors $\beta(z)$ and

$\beta(z_1)$, we must have colors $\beta(z) \in \{1,3\}$. It is impossible for two of $\beta(z) \in \{1,3\}$. So, we must recolor the colors $\beta(w)$ with $L(w) \setminus \{\beta(w_i) \bigcup \beta(x) \bigcup \beta(z)\}$. Thus, it is contradiction for suggestion.

Similarly, for the vertex z and w, we can deduce that the resulting coloring is an $(L,1)^* -$coloring, which is a contradiction. Therefore, the proof is complete.

Lemma 2.10 Let f be a 3-face by $(3,4,4^+)-$ face.

(i) If 3-vertex is a 3-poor vertex, then none of two 4-vertices is a 4-semi-poor vertex.

(ii) If a 3-vertex is a 3-poor vertex, then the neighbors of the third vertex not on e_f is $4^+ -$vertices.

(iii) If a 3-vertex is a 3-poor vertex, then at most one vertex of the neighbors of two 4-vertices is 3-vertex.

Proof: Let $f = [uu_1u_2] = (3,4,4^+) -$face and $N(u) = \{u_1, u_2, u_3\}$ and $N(u_i) = \{u_i', u_i''\}$ where $i = \{1,2\}$. We will prove the first (i). Let u be a 3-poor vertex. Suppose to the contrary that u_i is a 4-semi-poor vertex in which $i = \{1,2\}$. We note that u_i has a 4-vertex incident u_i' and u_i'' and then u_i'' is incident with u_3 Let $G' = \{u, u_1, u_2, u_1', u_1'', u_2', u_2'', u_3\}$. By minimality of G, suppose that $G - G'$ has an $(L,1)^* -$coloring of β. Without loss of generality, let $\beta(u) = \beta(u_2'') = \beta(u_1'') = 1$, $\beta(u_1) = \beta(u_2') = 2$ and $\beta(u_2) = \beta(u_1') = 3$. Since $|L(u_3)| \geq 1$, so we can assign the color $\beta(u_3)$ with 2 or 3. If we recolor $\beta(u)$ with 2, then we must assign the color $\beta(u_1)$ with 1. But $\beta(u_1'') = 1$. So, we must assign the color $\beta(u_1'')$ with 2 or 3. Here, by assumption, $u_1u_1' * u_1''$ must be a quadrilateral. So, $\beta(*)$ must be 2. Hence we must assign the color $\beta(u_1'')$ with 3. If we choose the colors $\beta(u_1'')$ with 3 and $\beta(u_3)$ with 2, we must assign the color $\beta(u_1')$ with 2. If we choose the colors $\beta(u_1'')$ with 2 and $\beta(u_3)$ with 3, then we must assign the color $\beta(u_1')$ with 3. If we recolor $\beta(u)$ with 3, then we must assign the color $\beta(u_1)$ with 2 or 1. If we choose $\beta(u_1)$ with 2 and $\beta(u_2)$ with 1, then we must assign the color $\beta(u_1')$ with 1 or 3 and $\beta(u_2'')$ with 2 or 3. If we choose the color $\beta(u_1')$ with 3, then we must assign the color $\beta(u_1'')$ with 2. Thus, it is contradiction by assumption. If we choose the color $\beta(u_1')$ with 1, then we must assign the colors $\beta(u_1'')$ with 3 and $\beta(u_3)$ with 2. If we choose the colors $\beta(u_2'')$ with 3 and $\beta(u_2')$ with 3, then it is contradiction by assumption. If we choose the color $\beta(u_2'')$ with 2 and $\beta(u_2')$ with 3, then it is contradiction.

We will prove the second (ii) and (iii) simultaneously. Here, since u_3 is incident with two 4-faces by Theorem 1.1 , so cannot be incident with any 4-faces. Thus, we have to know that it could be incident with $6^+ -$ faces. So, $d(u_3') \geq 4$ and $d(u_1'') = d(u_2'') = 3$. However, u_1' and u_2'' cannot be adjacent to 3-vertex because of u_1 and u_2 are not 4-poor vertices. Therefore, the proof is complete.

Lemma 2.11 Let u be a 3-vertex in a graph G. If u is a 3-semi poor vertex, then none of 4-face incident with u can be adjacent to

(i) a 4-poor vertex,

(ii) a 4-semi poor I vertex and

(iii) a 4-semi poor III vertex.

Proof: Let u be a 3-semi poor vertex in a graph G and $f_1 = [u_1uu_2x]$, $f_2 = [u_2uu_3y]$ and $f_3 = [u_3uu_1z]$ and then $N(u) = \{u_1, u_2, u_3\}$. We will prove first condition **(i)**. Suppose to the contrary that all of f_1, f_2 and f_3 are incident with 4-poor vertex. Firstly, we will prove a 4-poor vertex incident with f_1, f_2 and f_3. Without loss of generality, suppose that all of f_1, f_2 and f_3 are incident with a 4-poor vertex. Here, obviously we will assume that each of x, y and z is incident with a 4-poor vertex. We will consider a vertex by contraction of x, y and z. So, let $N(a)=\{a_1,a_2\}$. Continuously, we may construct each triangle incident with a such as u_1x_1x, u_2yy_1 and u_3zz_1. Then a_2 is incident with both 5-face and 6-face. Let $G'=\{u,u_1,u_2,u_3,a,a_1,a_2\}$. By minimality of G, suppose that $G-G'$ has an $(L,1)^*-$coloring of β. We will consider two cases.

Case (i). We may assume that $\beta(u_1)$, $\beta(u_2)$, and $\beta(u_3)$, are the same colors and $\beta(x)$, $\beta(y)$ and $\beta(z)$ are the same. So, we may assign the colors $\beta(u_1)$, $\beta(u_2)$ and $\beta(u_3)$ with 1 and then the colors $\beta(x)$, $\beta(y)$ and $\beta(z)$ with 2. Here, we must assign the color $\beta(u)$ with $L(u) \quad \{\beta(u_1),\beta(u_2),\beta(u_3)\}$ and we must assign the color $\beta(a_1)$ with 3. Evidently, 5-face is 3-coloring and 6-face is 2-coloring. So, we must assign the colors $\beta(a_2)$ with 1. Here, we will assign the color $\beta(u)$ with 3. Here, we must have all colors $\beta(x)$, $\beta(y)$ and $\beta(z)$ with 2. If we exchange the colors $\beta(u)$ and $\beta(u_1)$, we must recolor $\beta(u_2)$ with $L(u_2) \quad \{\beta(u_2')\}$, $\beta(u_3)$ with $L(u_3) \quad \{\beta(u_3')\}$ and $\beta(x_1)$, with $L(x_1) \quad \{\beta(x_1')\}$. Since $\beta(x_2)=1$, it must be $\beta(x_1')=1$. Now, we can have the color $\beta(x_1)$ with 2. It is contradiction. Moreover, since u_2 and u_3 are incident with 6-face and we have that 6-face is 2-coloring, they must be the colors $\beta(u_2')$ and $\beta(u_3')$ with 2. So, we must have the colors $\beta(u_2)$ and $\beta(u_3)$ with 3. It is contradiction.

Furthermore, since $|L(u)|=3$, we must assign the color $\beta(u)$ with 2. If we exchange the colors $\beta(u)$ and $\beta(u_1)$ we must recolor $\beta(u_2)$ with $L(u_2) \quad \{\beta(u_2')\}$ and $\beta(u_3)$ with $L(u_3) \quad \{\beta(u_3')\}$. So, we must have the colors $\beta(u_2)$ and $\beta(u_3)$ with 3. Then, we will exchange the colors $\beta(x)$ and $\beta(x_2)$. However, it is contradiction by assumption.

Case (ii). We may assume that $\beta(u_1)$, $\beta(u_2)$ and $\beta(u_3)$ are different. Evidently, we must have the colors $\beta(x)$, $\beta(y)$ and $\beta(z)$ are different. We may assume that the colors $\beta(u_1)$ with 1, $\beta(u_2)$ with 2 and $\beta(u_3)$ with 3. So, we must have the colors $\beta(x)$ with 3, $\beta(y)$ with 1 and $\beta(z)$ with 2 and then continuously we must have the colors $\beta(x_1)$ with 2, $\beta(y_1)$ with 3 and $\beta(z_1)$ with 1. If we assign the color $\beta(u)$ with 1, then we must recolor $\beta(u_1)$ with $L(u_1) \quad \{\beta(u_1')\}$. Thus, we must have the color $\beta(u_1)$ with distinct $\beta(u_1')$. Here, it is contradiction.

If we assign the color $\beta(u)$ with 2, then we must recolor $\beta(u_2)$ with $L(u_2) \quad \{\beta(u_2')\}$. Here, we must have the color $\beta(u_2)$ with distinct $\beta(u_2')$. However, it is contradiction. If we assign the color $\beta(u)$ with 3, then we must recolor $\beta(u_3)$ with $L(u_3) \quad \{\beta(u_3')\}$. Here, we must have the color $\beta(u_3)$ with distinct $\beta(u_3')$. However, it is contradiction.

Finally, for the condition **(ii)** and **(iii)** are similar as the proof of the condition **(i)**.

Therefore, the proof is complete. ■

Corollary 2.12 Suppose to v is a 3-semi-poor vertex in which $f_1=[vv_1xv_2]$, $f_2=[vv_2yv_3]$ and $f_3=[vv_3zv_1]$. If the three vertices of x, y and z are 3-semi-poor vertices, then the three vertices of v_1, v_2 and v_3 are 5^+-vertices.

Lemma 2.13 Let v be 3-vertex, $N(v)=\{v_1,v_2,v_3\}$ and $f=[vv_1v_2]$. If v is a 3-full-poor vertex in which v_1 and v_3 are incident with 5-face, then

(i) the three neighbors of v are 4^+-vertices (i.e., $d(N(u))\geq 4$) and

(ii) exactly the vertex v_1 is either a 4-poor vertex or a 5-poor vertex.

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Definition 2.14 (i) A vertex u is a $d(u)-$vertex incident with at most $n-$ triangles and others are any faces. Its vertex is called $T^{n\Delta}-$vertex.

Here, $|T^{n\Delta}|=$the number of $n-$triangles incident with a vertex

(ii) A vertex u is $d(u)-$vertex with $d(u)\geq 4$ in which u is incident with exactly $\left\lfloor \frac{d(u)}{2} \right\rfloor$ 3-faces and exactly $\left\lfloor \frac{d(u)}{4} \right\rfloor$4-faces. It is said to be a $T^{d(u)}-$vertex. Evidently, if $d(u)$ is odd, then every 4-face must be incident between two 3-faces.

Note that : If u is a 3-vertex incident with one 3-face and one 4-face or one 5-face, then another is one $8^{+}-$face. It is called $T^{1\Delta}-$vertex.

Lemma 2.15 Let u be $T^{d(u)}-$vertex in G.

Conditions: (i) If u is $T^{d(u)}-$vertex ($d(u)=3$), then it is incident with distinct one 3-face, one 4-face and one $8^{+}-$face. It is called a special $T^{3}-$vertex.

The following conditions:

Let u be $T^{d(u)}-$vertex in G with $d(u)\geq 4$.

(ii) If u is $T^{d(u)}-$vertex ($d(u)=4$), then it is incident with distinct two 3-faces, one 4-face and one $8^{+}-$face.

(iii) If u is $T^{d(u)}-$vertex (where $d(u)=5$), then it is incident with distinct two 3-faces, one 4-face, and then others are $5^{+}-$faces.

(iv) For $d(u)\geq 6,$ if u is a $T^{d(u)}-$vertex and $d(u)$ is odd, then it is incident with at most two $5^{+}-$faces and others are incident with at most $\left\lceil \frac{d(u)-1}{4} \right\rceil -1$ $8^{+}-$faces.

(v) For $d(u)\geq 6,$ if u is a $T^{d(u)}-$vertex and $d(u)$ is even, then it is incident with at most $\left\lceil \frac{d(u)}{4} \right\rceil$ $8^{+}-$faces.

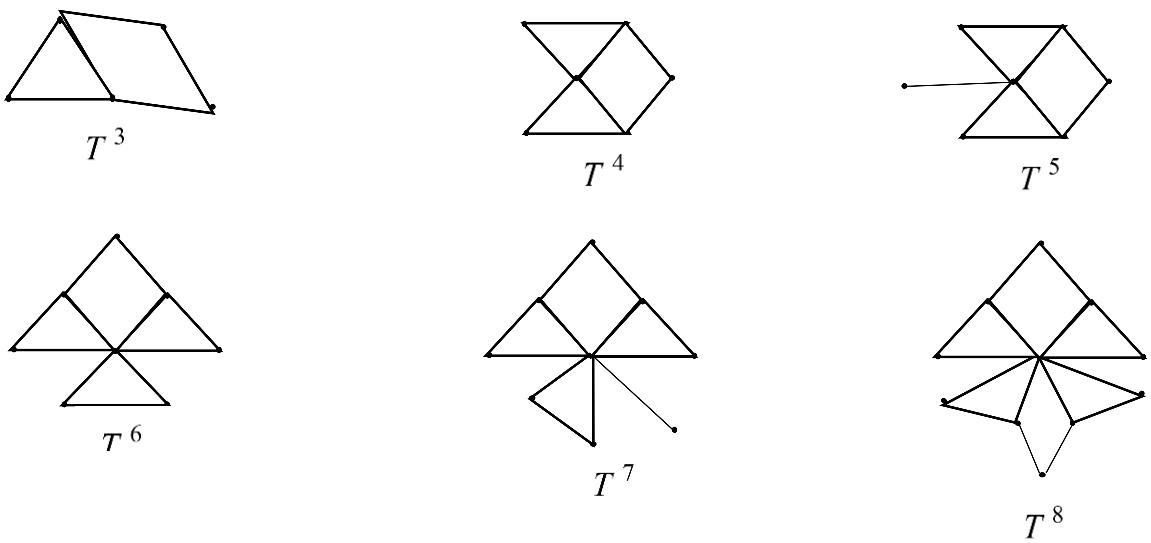

Figure 7:

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Corollary 2.16 If u is a $T^{d(u)}-$vertex $(d(u)\ge 7,\ d(u)=4n+3,\ n=1,2,\dots)$ in which there are incident with at most $\left\lfloor \frac{d(u)}{2} \right\rfloor$ 3-faces and at most $\left\lfloor \frac{d(u)}{4} \right\rfloor$ 4-faces, then there are at most two $5^{+}-$faces and $(\frac{d(u)}{4}-\frac{3}{4})$ $8^{+}-$faces.

Corollary 2.17 If u is a $T^{d(u)}-$vertex $(d(u)\ge 9,\ d(u)=4n+5,\ n=1,2,\dots)$ in which there are incident with at most $\left\lfloor \frac{d(u)}{2} \right\rfloor$ 3-faces and at most $\left\lfloor \frac{d(u)}{4} \right\rfloor$ 4-faces, then there are at most two $5^{+}-$faces and $(\frac{d(u)}{4}-\frac{5}{4})$ $8^{+}-$faces.

3. DISCHARGING PROCESS

We now apply a discharging procedure to reach a contradiction. We first define the initial charge function ch on the vertices and faces of G by letting $ch(v)=ad(v)-2b$ if $v\in V(G)$ and $ch(f)=(b-a)d(f)-2b$, $f\in F(G)$. We note $a=\frac{3}{2}$ and $b=\frac{7}{2}$ so that we get the initial function $ch(v)=\frac{3}{2}d(v)-7$ if $v\in V(G)$ and $ch(f)=2d(f)-7,\quad f\in F(G)$. It follows from Euler's formula $|V(G)|-|E(G)|+|F(G)|=2$ and the relation

$$\sum_{v\in V(G)} d(v)=\sum_{f\in F(G)} d(f)=2|E(G)|$$

so that the total sum of initial function of the vertices and faces is equal to

$$\begin{aligned}\sum_{v\in V(G)} ch(v)+\sum_{f\in F(G)} ch(f)&=\sum_{v\in V(G)}(\frac{3}{2}d(v)-7)+\sum_{f\in F(G)}(2d(f)-7)\\ &=\frac{3}{2}(2|E(G)|)-7|V(G)|+2(2|E(G)|)-7|F(G)|\\ &=-7(|V(G)|+|F(G)|-|E(G)|)\ =-14\end{aligned}$$

Since any discharging procedure preserves the total charge of G, if we can define suitable discharging rules to change the initial charge function ch to the final charge function ch' on $V\bigcup F$ such that $ch'(x)\ge 0$ for all $x\in V\bigcup F$, then

$$0\le \sum_{x\in V\bigcup F} ch'(x)=\sum_{x\in V\bigcup F} ch(x)=-14,$$

a contradiction completing the proof of Theorem 1.1 when G is 2-connected.

Proof of Theorem 1.1

Since G is 2-connected, G has no adjacent 3-faces or 7-cycles and $\delta(G)\ge 3$, the following Lemma is obvious.

Lemma 3.1

(i) In G, there is no adjacent 3-faces.

(ii) In G, there is a 4-face adjacent to at most two 3-faces. Moreover, when a 4-face is adjacent to at least one 3-face, the 4-face can be adjacent to no 4-face except v is a 3-poor vertex.

(iii) In G, there is a 4-face adjacent to at least one 4-face.

(iv) In G, there is a 5-face adjacent to at most one 3-face and no adjacent to any 4-face.

(v) In G, there is no 6-face adjacent to a 3-face.

We will introduce the discharging rules:

R 1. Charge from a 4^+ − face f

R 1.1. If $d(f)=4$, then f sends $\frac{1}{4}$ to each incident vertex.

R 1.2. If $d(f)=5$, then f sends $\frac{3}{5}$ to each incident vertex.

R 1.3. If $d(f)=6$, then f sends $\frac{5}{6}$ to each incident vertex.

R 1.4. If $d(f)\geq 8$, then f sends $\frac{9}{8}$ to each incident vertex.

R 2. Charge to a 3-face $f=[v_1v_2v_3]$ where $d(v_1)\leq d(v_2)\leq d(v_3)$.

R 2.1. Suppose to v is a 4-light vertex.

Let $f=[v_1v_2v]=(5^+,3,4)$ − face. Then v gets $\frac{9}{8}$ from each 8^+ − face and $\frac{1}{4}$ from 4-face and it sends $\frac{3}{2}$ to f. Then v_2 gets $\frac{10}{8}$ from 8^+ − face and $\frac{5}{4}$ from f. After that v_1 gets $\frac{9}{8}$ from 8^+ − face and sends $\frac{13}{16}$ to f.

R 3. Suppose to v is a poor vertex in which $f=[v_1v_2v_3]$ with $d(v_1)\leq d(v_2)\leq d(v_3)$.

R 3.1. Let $d(v_1)=3$ and v_1 be a 3-poor vertex. Then v_1 gets $\frac{1}{2}$ from each 4-face and f sends $\frac{3}{2}$ to v_1.

R 3.2. Let $d(v_2)=4$ and v_2 be a 4-poor vertex. v_2 gets $\frac{3}{5}$ from 5-face and $\frac{5}{6}$ from 6-face and f gets $\frac{1}{3}$ from v_2.

R 3.3. Let $d(v_3)=5$ and v_3 be a 5-poor vertex. v_3 gets $\frac{3}{5}$ from 5-face, $\frac{5}{6}$ from 6^+ − face and $\frac{5}{6}$ from 5^+ − face and then f gets $\frac{8}{3}$ from v_3.

R 4. Suppose to v be a 3-semi-poor vertex in which $f_1=[vv_1xv_2]$, $f_2=[vv_2yv_3]$ and $f_3=[vv_3zv_1]$ with $d(v)\leq d(v_i)$ where $i\in\{1,2,3\}$.

R 4.1. Let $d(v)=3$ and v be a 3-semi-poor vertex. Then v gets $\frac{5}{6}$ from each 4-face.

R 4.2. Let $d(x)=d(y)=d(z)=3$ and they be 3-semi-poor vertices. So, v_i a 5^+ − vertex where $i\in\{1,2,3\}$. Then v gets $\frac{1}{2}$ from each 4-face and $\frac{1}{3}$ from each 5^+ − vertex and 4-face sends $\frac{1}{6}$ to other vertices not 3-semi-poor vertices. Moreover, x, y and z are like as v.

R 5. Suppose to v_1 be a 3-full-poor vertex in which $f=[v_1v_2v_3]$ with $d(v_1)\leq d(v_2)\leq d(v_3)$.

Then v_1 gets $\frac{3}{5}$ from 5-face and $\frac{18}{8}$ from 8^+-face and v_1 sends $\frac{7}{20}$ to f. Moreover, 8^+-face sends $\frac{27}{28}$ to other vertices.

R 6. Suppose to v be a 4-semi-poor vertex in which $f_1=[vv_1v_2]$, $f_3=[vv_3xv_4]$ and f_2 and f_4 are 8^+-faces with $d(v_1)=d(v_4)=3$.

R 6.1 Let v be a 4-semi-poor I vertex. Then v gets $\frac{1}{3}$ from f_3 and $\frac{9}{8}$ from 8^+-face and it sends $\frac{3}{2}$ to f_1.

R 6.1.1 For $d(v_1)=d(v_4)=3$, v_1 gets $\frac{9}{8}$ from f_1, $\frac{1}{4}$ from 4-face and $\frac{9}{8}$ from 8^+-face and then v_4 gets $\frac{2}{3}$ from f_3 and $\frac{9}{8}$ from 8^+-face.

R 6.2 Let v be a 4-semi-poor II vertex. Then v gets $\frac{1}{4}$ from f_3 and $\frac{9}{8}$ from 8^+-face and it sends $\frac{3}{2}$ to f_1.

R 6.2.1 For $d(v_1)=3$, v_1 gets $\frac{9}{8}$ from f_1, $\frac{1}{4}$ from 4-face and $\frac{9}{8}$ from 8^+-face.

R 6.2.2 For $d(v_4)=3$, if the outer neighbor of v_4 is 4-semi-poor vertex, then v_4 gets $\frac{3}{4}$ from f_3, $\frac{2}{3}$ from 4-face and $\frac{9}{8}$ from 8^+-face. If the outer neighbor of v_4 is not 4-semi-poor vertex, then v_4 gets $\frac{3}{4}$ from f_3 and $\frac{1}{4}$ from 4-face and $\frac{9}{8}$ from 8^+-face.

R 6.3 Let v be a 4-semi-poor III vertex. Then v gets $\frac{1}{3}$ from f_3 and $\frac{9}{8}$ from 8^+-face and it sends $\frac{3}{2}$ to f_1.

R 6.3.1 For $d(v_1)=d(v_4)=3$, v_1 gets $\frac{7}{8}$ from f_1, $\frac{3}{5}$ from 5-face and $\frac{9}{8}$ from 8^+-face and then v_4 gets $\frac{2}{3}$ from f_3 and $\frac{9}{8}$ from 8^+-face.

R 6.4 Let v be a 4-semi-poor IV vertex. Then v gets $\frac{1}{4}$ from f_3 and $\frac{9}{8}$ from 8^+-face and it sends $\frac{3}{2}$ to f_1.

R 6.4.1 For $d(v_1)=3$, v_1 gets $\frac{7}{8}$ from f_1, $\frac{3}{5}$ from 5-face and $\frac{9}{8}$ from 8^+- face.

R 6.4.2 For $d(v_4)=3$, if the outer neighbor of v_4 is 4-semi-poor vertex, then v_4 gets $\frac{3}{4}$ from f_3, $\frac{2}{3}$ from 4-face and $\frac{9}{8}$ from 8^+-face. If the outer neighbor of v_4 is not 4-semi-poor vertex, then v_4 gets $\frac{2}{3}$ from f_3 and $\frac{1}{4}$ from 4-face and $\frac{9}{8}$ from 8^+-face.

R 7. Suppose to v be a 4-full-poor vertex in which $f_1=[vv_1xv_2]$, $f_3=[vv_3yv_4]$ and f_2 and f_4 are 8^+-faces with $d(v_1)=d(v_4)=3$.

R 7.1 Let v be a 4-full-poor I vertex. Then v gets $\frac{9}{8}$ from each 8^+-face and it sends $\frac{2}{5}$ to v_1 and v_4.

R 7.1.1 For $d(v_1)=d(v_4)=3$, both v_1 and v_4 get $\frac{1}{2}$ from 4-face and $\frac{9}{8}$ from 8^+-face and then they get $\frac{2}{5}$ from v. Moreover, f_1 and f_2 send $\frac{1}{2}$ to 3-vertex and $\frac{1}{8}$ to 4^+-vertex.

R 7.2 Let v be a 4-full-poor II vertex and v_1 is incident with 3-face and v_4 is incident with 4-face. Then v gets $\frac{9}{8}$ from each 8^+- face and it sends $\frac{2}{5}$ to v_1 and $\frac{1}{5}$ to v_4.

R 7.2.1 For $d(v_1)=3$, v_1 gets $\frac{1}{2}$ from 4-face and $\frac{9}{8}$ from 8^+-face and then it gets $\frac{2}{5}$ from v.

R 7.2.2 For $d(v_4)=3$, if the outer neighbor of v_4 is 4-semi-poor vertex, then v_4 gets $\frac{1}{2}$ from f_3 $\frac{2}{3}$ from 4-face and $\frac{9}{8}$ from 8^+- face and then gets $\frac{1}{5}$ from v. If the outer neighbor of v_4 is not 4-semi-poor vertex, then v_4 gets $\frac{1}{2}$ from f_3 and $\frac{1}{4}$ from 4-face and $\frac{9}{8}$ from 8^+-face and then $\frac{1}{5}$ from v.

R 7.3 Let v be a 4-full-poor III vertex. Then v gets $\frac{9}{8}$ from each 8^+-face and it sends $\frac{2}{5}$ to both v_1 and v_4.

R 7.3.1 For $d(v_1)=d(v_4)=3$, if the outer neighbors of v_1 and v_4 is 4-semi-poor vertices, then both of v_1 and v_4 get 1 from each 4-face and $\frac{9}{8}$ from 8^+-face and then get $\frac{2}{5}$ from v. If the outer neighbors of v_1 and v_4 are not 4-semi-poor vertices, then v_1 and v_4 get 1 from f_1 and f_3 and $\frac{1}{4}$ from 4-face and $\frac{9}{8}$ from 8^+- face and then $\frac{2}{5}$ from v.

R 8. Suppose to v is $T^{d(v)}-$vertex.

We deduce induction for $d(v)\geq 3$.

R 8.1 T^3-vertex.

Let $f=[vv_1v_2]$ and v be 3-vertex incident with 4-face and 8^+-face. If v is a T^3 vertex, then v gets $\frac{9}{8}$ from 8^+-face and $\frac{1}{4}$ from 4-face. Then f sends $\frac{9}{8}$ to v.

R 8.2 T^4-vertex.

If v is T^4-vertex incident with one 4-face and one 8^+-face, then v gets $\frac{10}{8}$ from 8^+-face and $\frac{1}{4}$ from 4-face and then v sends $\frac{2}{8}$ to each 3-face.

R 8.3. T^5-vertex

Let $f_1=[vv_1v_2]$ and $f_3=[vv_3v_4]$. v gets $\frac{3}{5}$ from each 5^+-face and $\frac{1}{4}$ from 4-face. Then v sends $\frac{7}{8}$ to each 3-face.

R 8.4. $T^{d(v)}$ – vertex

R 8.4.1 Let v be a $T^{d(v)}$ – vertex such that n is even and $n \geq 6$. v gets $\frac{3}{8}$ from each 8^+ – face and $\frac{1}{4}$ from 4-face. In general , v sends $\frac{53d(v)-224}{16d(v)}$ to each 3-face.

R 8.4.2 Let v be a $T^{d(v)}$ – vertex such that $d(v)$ is odd and $d(v) \geq 7$. Here v is incident with $(\frac{d(v)}{4} - \frac{3}{4})$ 8^+ – face where $d(v) = 4r+3$, $r = 1,2,\ldots,n$ and $d(v) \geq 7$ and incident with $\left\lfloor \frac{d(v)}{2} \right\rfloor$ 3-face and two 5^+ – face.

Then v gets $\frac{3}{8}$ from each 8^+ – face, $\frac{1}{4}$ from 4-face and $\frac{3}{5}$ from each 5^+ – face.

In general for $d(v) = 4n+3$, $n = 1,2,\ldots,$ and $d(v) \geq 7$, v sends $\frac{52d(v)-194}{16d(v)}$ to each 3-face.

R 8.4.3 Let v be a $T^{d(v)}$ – vertex such that $d(v)$ is odd and $d(v) \geq 9$. Here v is incident with $(\frac{d(v)}{4} - \frac{5}{4})$ 8^+ – face where $d(v) = 4n+5$, $n = 1,2,\ldots$ and $d(v) \geq 9$ and incident with $\left\lfloor \frac{d(v)}{2} \right\rfloor$ 3-face and two 5^+ – face.

Then v gets $\frac{3}{8}$ from each 8^+ – face, $\frac{1}{4}$ from each 4-face and $\frac{3}{5}$ from each 5^+ – face.

In general for $d(v) = 4n+5$, $n = 1,2,\ldots,$ and $d(v) \geq 9$, v sends $(\frac{52d(v)-202}{16d(v)})$ to 3-face.

R 9. For $d(v) \geq 4$, if v is incident with 3-face, 4-face, 6^+ – face and 8^+ – face, then v gets $\frac{1}{4}$ from 4-face, $\frac{5}{6}$ from 6^+ – face and $\frac{9}{8}$ from 8^+ – face and sends 1 to 3-face.

R 10. Otherwise, if v is not a poor vertex in which $f = [v_1, v_2, v_3] = (3,4,5)$ – face, then f gets 1 from 4-vertex and $\frac{3}{2}$ from 5-vertex and then it sends $\frac{9}{8}$ to v_1.

It remains to show that the resulting final charge ch' is satisfied with $ch' \geq 0$ for all $x \in V \bigcup F$. Let $v \in V(G)$ and $f \in F(G)$. The proof can be completed with $d(x)$ for all $x \in V \bigcup F$. Let $v \in V(G)$ and $f \in F(G)$. Since $d(v) \geq 3$. If $d(v) = 4$, by **R 1** and **R 2,** then v is a 4-light vertex with $f = (3,4,5^+)$ – face. So, $ch'(v) = ch(v) + 2 \times \frac{9}{8} + \frac{1}{4} - \frac{3}{2} = \frac{3}{2} \times 4 - 7 + 2 \times \frac{9}{8} + \frac{1}{4} - \frac{3}{2} = 0$ by **R 2.1**. Continuously, if $d(v) = 3$ by **R 2.1** and **R 5**, then $f = (3,4,5^+)$ – face and the 3-vertex is 3-full-poor vertex. So, $ch'(v) = ch(v) + \frac{10}{8} + \frac{5}{4} = 0$ by **R 2.1** and $ch'(v) = ch(v) + \frac{10}{8} + \frac{5}{4} + \frac{3}{5} > 0$ **R 5**.

If $f = [v_1 v_2 v_3] = (3,4,5)$ by **R 1** and **R 3** and by Lemma 2.8, then v_1, v_2 and v_3 are 3-poor, 4-poor and 5-poor vertices. So, for $ch'(v) = ch(v) + 2 \times \frac{1}{2} + \frac{3}{2} = 0$ by **R 3.1**. And then for $d(v) = 4$,

$ch'(v)=ch(v)+\frac{3}{5}+\frac{5}{6}-\frac{1}{3}=-1+\frac{3}{5}+\frac{5}{6}-\frac{1}{3}\geq 0$ **R 3.2**. Moreover, for $d(v)=5$, $ch'(v)=ch(v)+\frac{3}{5}+2\times\frac{5}{6}-\frac{8}{3}=\frac{1}{2}+\frac{3}{5}+2\times\frac{5}{6}-\frac{8}{3}\geq 0$ **R 3.3**. If $d(v)=3$ and $f_1=[vv_1xv_2]$, $f_2=[vv_2yv_3]$ and $f_3=[vv_3zv_1]$, then v is a 3-semi-poor vertex by **R 1** and **R 4**. So, we have $ch'(v)=ch(v)+3\times\frac{5}{6}=\frac{3}{2}\times 3-7+3\times\frac{5}{6}=0$ by **R 4.1**. By Corollary 2.12 if $d(x)=d(y)=d(z)=3$ and they are 3-semi-poor vertices, then $d(v_i)\geq 5$. So, $ch'(v)=ch(v)+3\times\frac{1}{2}+3\times\frac{1}{3}=-\frac{5}{2}+\frac{5}{2}=0$ by **R 4.2**. If $d(v)=3$ and $f=[vv_1v_2]=(3,4,4^+)$ and $N(v)=\{v_1,v_2,v_3\}$ by **R 1** and **R 5** and by Lemma 2.13, then v is a 3-full-poor vertex. So, $ch'(v)=ch(v)+\frac{3}{5}+\frac{18}{8}-\frac{7}{20}=-\frac{5}{2}+\frac{5}{2}=0$ by **R 5**. Then, if v_1 is a 4-poor vertex, then v_2 is incident with 4-face, 6^+-face and 8^+-face. So, for $d(v)\geq 4$, $ch'(v)=ch(v)+\frac{1}{4}+\frac{5}{6}+\frac{27}{28}-1\geq 0$ by **R 9** and **R 5**. Here, for 3-face, $ch'(f)=ch(f)+\frac{1}{3}+\frac{7}{20}+1>0$ **R 3.2** and **R 5** and **R 9**.

For $d(v)=4$, if $f_1=[vv_1v_2]$, $f_3=[vv_3xv_4]$ and f_2 and f_4 are 8^+-faces with $d(v_1)=d(v_4)=3$, then v is a 4-semi-poor vertex by **R 1** and **R 6**. If v is a 4-semi-poor vertex I, then $ch'(v)=ch(v)+\frac{1}{3}+2\times\frac{9}{8}-\frac{3}{2}=-1+\frac{1}{3}+\frac{9}{4}-\frac{3}{2}>0$ by **R 6.1**. For $d(v_1)=3$, we must have $d(v_2)\geq 4$. So, $ch'(v_1)=ch(v_1)+\frac{9}{8}+\frac{1}{4}+\frac{9}{8}=0$ by **R 6.1.1** and **R 9**. Then $f=[vv_1v_2]$, $ch'(f)=ch(f)+\frac{3}{2}+1-\frac{9}{8}>0$ by **R 6.1**, **R 6.1.1** and **R 9**. For $d(v_4)=3$, if v_4 is incident with $f=(3,4,5)-$face, then $ch'(v_4)=ch(v_4)+\frac{2}{3}+\frac{9}{8}+\frac{9}{8}>0$ by **R 6.1.1** and **R 10**. If v is a 4-semi-poor vertex II, then $ch'(v)=ch(v)+\frac{1}{4}+2\times\frac{9}{8}-\frac{3}{2}=-1+\frac{1}{3}+\frac{9}{4}-\frac{3}{2}=0$ by **R 6.2**. For $d(v_4)=3$, if the outer neighbor of v_4 is 4-semi-poor vertex, then $ch'(v_4)=ch(v_4)+\frac{2}{3}+\frac{3}{4}+\frac{9}{8}\geq 0$ by **R 6.2.2**. For $d(v_4)=3$, if the outer neighbor of v_4 is 4-full-poor vertex, then $ch'(v_4)=ch(v_4)+\frac{1}{4}+\frac{3}{4}+\frac{2}{5}+\frac{9}{8}\geq 0$ by **R 6.2.2** and **R 7.1**.

If v is a 4-semi-poor vertex III, then $ch'(v)=ch(v)+\frac{1}{3}+2\times\frac{9}{8}-\frac{3}{2}=-1+\frac{1}{3}+\frac{9}{4}-\frac{3}{2}>0$ by **R 6.3**. For $d(v_1)=3$, we must have $d(v_2)\geq 4$. So, $ch'(v_1)=ch(v_1)+\frac{7}{8}+\frac{3}{5}+\frac{9}{8}>0$ by **R 6.3.1** and **R 9**. Then $f=[vv_1v_2]$, $ch'(f)=ch(f)+\frac{3}{2}+1-\frac{7}{8}>0$ by **R 6.3**, **R 6.3.1** and **R 10**. For $d(v_4)=3$, if v_4 is incident with $f=(3,4,5)-$face, then $ch'(v_4)=ch(v_4)+\frac{2}{3}+\frac{9}{8}+\frac{9}{8}>0$ by **R 6.3.1** and **R 10**.

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If v is a 4-semi-poor vertex IV, then $ch'(v)=ch(v)+\frac{1}{4}+2\times\frac{9}{8}-\frac{3}{2}=-1+\frac{1}{3}+\frac{9}{4}-\frac{3}{2}=0$ by **R 6.4**. For $d(v_4)=3,$ if the outer neighbor of v_4 is 4-semi-poor vertex, then $ch'(v_4)=ch(v_4)+\frac{2}{3}+\frac{3}{4}+\frac{9}{8}\geq 0$ by **R 6.4.2**. For $d(v_4)=3,$ if the outer neighbor of v_4 is 4-full-poor vertex, then $ch'(v_4)=ch(v_4)+\frac{1}{4}+\frac{3}{4}+\frac{2}{5}+\frac{9}{8}\geq 0$ by **R 6.4.2** and **R 7.1**.

For $d(v)=4,$ if $f_1=[vv_1xv_2],$ $f_3=[vv_3yv_4]$ and f_2 and f_4 are 8^+-faces with $d(v_1)=d(v_4)=3,$ then v is a 4-full-poor vertex by **R 1** and **R 7**. If v is a 4-full-poor vertex I, then $ch'(v)=ch(v)+\frac{1}{8}+2\times\frac{9}{8}-2\times\frac{2}{5}=-1+\frac{9}{4}-\frac{4}{5}>0$ by **R 7.1**. For $d(v_1)=d(v_4)=3,$ if v_1 and v_4 are incident with $f=(3,4,5)$, then $ch'(v)=ch(v)+\frac{1}{2}+\frac{2}{5}+\frac{9}{8}+\frac{9}{8}>0$ by **R 7.1.1** and **R 11** (where v is represented by v_1 and v_4). If v is a 4-full-poor vertex II, then $ch'(v)=ch(v)+\frac{1}{8}+2\times\frac{9}{8}-\frac{2}{5}-\frac{1}{5}=-1+\frac{1}{8}+\frac{9}{4}-\frac{3}{5}>0$ by **R 7.2**. For $d(v_4)=3,$ if the outer neighbor of v_4 is 4-semi-poor vertex, then $ch'(v_4)=ch(v_4)+\frac{1}{2}+\frac{1}{5}+\frac{9}{8}+\frac{2}{3}=0$ by **R 7.2.2** and **R 6.1**. For $d(v_4)=3,$ if the outer neighbor of v_4 is 4-full-poor vertex, then $ch'(v_4)=ch(v_4)+\frac{1}{4}+\frac{1}{2}+\frac{1}{5}+\frac{2}{5}+\frac{9}{8}=0$ by **R 7.2.2** and **R 7.1**.

For $d(v)=3,$ by **R 1** and **R 8**, if v is incident with 3-face, 4-face and 8^+-face, then v is a T^3-vertex. Let $f=[vv_1v_2]=(3,4,4^+)-$face. Here, v is T^3-vertex and we can get v_1 is a 4-semi-poor vertex and $v_2\geq 4$ and so $ch'(v)=ch(v)+\frac{1}{4}+\frac{9}{8}+\frac{9}{8}=0$ by **R 8.1**, **R 6** and **R 9**. Then $ch'(f)=ch(f)+\frac{3}{2}+1-\frac{9}{8}>0$ by **R 8.1**, **R 6** and **R 9**.

For $d(v)=4,$ by **R 1** and **R 8**, if v is incident with two 3-faces, one 4-face and one 8^+-face, then v is a T^4- vertex. Let $f_1=[vv_1v_2]$ and $f_3=[vv_3v_4],$ f_2 be 4-face and f_4 is 8^+-face. So, $ch'(v)=ch(v)+\frac{1}{4}+\frac{10}{8}-2\times\frac{2}{8}=0$ by **R 8.2**. Let $f_1=f_3=(3,4,5)$. If v is a T^4-vertex, then $ch'(f)=ch(f)+\frac{2}{8}+\frac{3}{2}-\frac{9}{8}<0$ by **R 8.2**, **R 10** or $ch'(f)=ch(f)+\frac{2}{8}+\frac{3}{2}-\frac{3}{2}<0$ by **R 8.2**, **R 3.1**. So, it is impossible that T^4- vertex is adjacent to 3-vertex. ■

Lemma 3.2 Let $f_1=[vv_1v_2]$ and $f_3=[vv_3v_4],$ f_2 be 4-face and f_4 is 8^+-face. If v is a T^4-vertex, then the neighbor vertices of v are 4^+-vertex.

For $d(v)=5,$ by **R 1** and **R 8**, if v is incident with two 3-faces, one 4-face and two 5^+-face, then v is a T^5-vertex. Let $f_1=[vv_1v_2]$ and $f_3=[vv_3v_4],$ f_2 be 4-face and f_4 and f_5 are 5^+-faces. So, $ch'(v)=ch(v)+\frac{1}{4}+2\times\frac{3}{5}-2\times\frac{7}{8}=0$ by **R 8.3**. For f_1 and f_2, if v is a T^5-vertex, then $ch'(f)=ch(f)+\frac{7}{8}+1-\frac{3}{2}<0$ by **R 8.3**, **R 9** and **R 3.1** or $ch'(f)=ch(f)+\frac{7}{8}+\frac{3}{2}-\frac{3}{2}<0$ by **R 8.2**, **R 3.1** and **R 10**. So, it is impossible that T^5-vertex is adjacent to 3-poor vertex. Then $ch'(f)=ch(f)+\frac{7}{8}+\frac{3}{2}-\frac{9}{8}>0$ by **R 8.2**, **R 10** and $ch'(f)=ch(f)+\frac{7}{8}+1-\frac{9}{8}<0$ by **R 8.2, R 9**. Therefore, if v is T^5-vertex adjacent to T^3-vertex, then $f=(5,3,5^+)-$face.

Lemma 3.3 In G, let v be a T^5-vertex in which $f_1=[vv_1v_2]$ and $f_3=[vv_3v_4]$, f_2 be 4-face and f_5 be 5^+-faces. If a T^5-vertex is adjacent to T^3-vertex, then $f_1=f_2=(5,3,5^+)-$face.

Moreover, if v is a $T^{d(v)}-$vertex, where $d(v)\geq 6$ and $d(v)$ is even, by Lemma 2.15, then v is incident at most $\left\lfloor\frac{d(v)}{2}\right\rfloor$ 3-faces, at most $\left\lfloor\frac{d(v)}{4}\right\rfloor$ 4-faces and at most $\left\lceil\frac{d(v)}{4}\right\rceil 8^+-$faces. So, by **R 1** and R 8,

$$
\begin{aligned}
ch'(v)&\geq ch(v)+\frac{3}{8}(\left\lceil\frac{d(v)}{4}\right\rceil)+\frac{1}{4}(\left\lfloor\frac{d(v)}{4}\right\rfloor)-\frac{53d(v)-224}{16d(v)}\left\lfloor\frac{d(v)}{2}\right\rfloor\\
&=\frac{3}{2}d(v)-7+\left\lceil\frac{3d(v)}{32}\right\rceil+\left\lfloor\frac{2d(v)}{32}\right\rfloor-\frac{53d(v)-224}{16d(v)}\left\lfloor\frac{d(v)}{2}\right\rfloor\\
&=\frac{53d(v)-224}{32}-\frac{53d(v)-224}{16d(v)}\left\lfloor\frac{d(v)}{2}\right\rfloor\geq 0
\end{aligned}
$$

by **R 8.4.1**.

If v is a $T^{d(v)}-$vertex $(d(v)\geq 7,\ d(v)=4n+3$, where $n=1,2,\dots)$ by **R 8.4.2** and by Corollary 2.16, then

$$
\begin{aligned}
ch'(v)&\geq ch(v)+\frac{3}{8}(\frac{d(v)}{4}-\frac{3}{4})+\frac{1}{4}(\left\lfloor\frac{d(v)}{4}\right\rfloor)+2\times\frac{3}{5}-(\frac{52d(v)-194}{16d(v)})\left\lfloor\frac{d(v)}{2}\right\rfloor\\
&=\frac{3}{2}d(v)-7+\frac{3d(v)}{32}+\left\lfloor\frac{d(v)}{16}\right\rfloor+\frac{6}{5}-\frac{9}{32}-\frac{52d(v)-194}{16d(v)}\left\lfloor\frac{d(v)}{2}\right\rfloor\\
&=\frac{51d(v)}{32}+\left\lfloor\frac{d(v)}{16}\right\rfloor-\frac{973}{160}-\frac{52d(v)-194}{16d(v)}\left\lfloor\frac{d(v)}{2}\right\rfloor\\
&\leq\frac{265d(v)-973}{160}-\frac{52d(v)-194}{32}\\
&=\frac{265d(v)-973}{160}-\frac{260d(v)-970}{160}\\
&>0
\end{aligned}
$$

If v is a $T^{d(v)}-$vertex $(d(v)\geq 9,\ d(v)=4n+5$, where $n=1,2,\dots)$ by **R 8.4.3** and by Corollary 2.17, then

$$ch'(v) \geq ch(v) + \frac{3}{8}(\frac{d(v)}{4} - \frac{5}{4}) + \frac{1}{4}(\left\lfloor \frac{d(v)}{4} \right\rfloor) + 2 \times \frac{3}{5} - (\frac{52d(v)-202}{16d(v)})\left\lfloor \frac{d(v)}{2} \right\rfloor$$
$$= \frac{3}{2}d(v) - 7 + \frac{3d(v)}{32} + \left\lfloor \frac{d(v)}{16} \right\rfloor + \frac{6}{5} - \frac{15}{32} - \frac{52d(v)-202}{16d(v)}\left\lfloor \frac{d(v)}{2} \right\rfloor$$
$$= \frac{51d(v)}{32} + \left\lfloor \frac{d(v)}{16} \right\rfloor - \frac{1018}{160} - \frac{52d(v)-202}{16d(v)}\left\lfloor \frac{d(v)}{2} \right\rfloor$$
$$\leq \frac{265d(v)-1018}{160} - \frac{52d(v)-202}{32}$$
$$= \frac{265d(v)-1018}{160} - \frac{260d(v)-1010}{160} > 0$$

If v is a 4-light vertex, then $f=[v_1v_2v]=(3,3,4)-$face by **R1** and **R2.1** and **R 5**. If v_1 and v_2 are 3-full-poor vertices, then $ch'(f)=ch(f)+1+\frac{3}{4}+\frac{7}{20}=2d(f)-7+\frac{22}{20}\geq 0$. By Lemma 2.9, when $d(f)=4$, f sends $\frac{1}{4}$ to each 4-light vertex. $ch'(f)=ch(f)-4\times\frac{1}{4}=0$ by **R 2.1** and **R 1**. Suppose $d(f)=3$ with $f=[v_1v_2v_3]=(3,4,5)-$face. By Lemma 2.8 and **R 3**, if v_1, v_2 and v_3 are poor vertices, then $ch'(f)=ch(f)+\frac{1}{3}+\frac{8}{3}-\frac{3}{2}=2d(f)-7+\frac{3}{2}>0$ by **R 3.1**, **R 3.2** and **R 3.3**. By **R 10**, if v_1, v_2 and v_3 are not poor vertices, then $ch'(f)=ch(f)+\frac{3}{2}+1-\frac{9}{8}=2d(f)-7+\frac{11}{8}>0$. For $d(f)=4$, by Lemma 2.11, $ch'(f)=ch(f)-\frac{5}{6}-\frac{1}{3}=2d(f)-7-\frac{7}{6}<0$ by **R 3.2**, **R 4.1** and **R 6.1**. So, Lemma 2.11 is true.

We have that G is simple, has neither adjacent triangles nor 7-cycles and $\delta(G)\geq 3$, the following lemma is obvious. This completes the proof of Theorem 1.1.

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