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ON APPROXIMATION BY SOME INTEGRAL MODIFICATION OF JAKIMOVSKI-LEVIATAN OPERATORS

Nandita Gupta[1]
[1]Government Polytechnic, Mahasamund , Chhattisgarh, India.
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ABSTRACT

In the present paper we propose an integral modification of Jakimovski- Leviatan operators involving Appell Polynomials. Using Korovkin Theorem we obtain the approximation properties of these operators. We compute an estimate of the order of approximation of a continuous functions by means of the operator via first order modulus of continuity. We also give an asymptotic estimate through Voronovskaja - type result for these operators.

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Key Words- Linear positive operators, Appell Polynomials, Order of approximation, Modulus of continuity, Voronovskaja - type theorem .

1. INTRODUCTION

Jakimovski and Leviatan [3] introduced a new type of operators Pn by using Appell polynomials as follows,

$$P_n(f;x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right) \qquad (1.1)$$

for f ∈ E[0, ∞).B. Wood proved in [6] that the operators Pn are positive on [0, ∞)

an

if and only if ≥ 0, n ∈ N ∈ a_n

g(1)

Recall that Appell polynomials are polynomials defined as follws: Let

$$g(u) = \sum_{n=0}^{\infty} a_n u^n \, , g(1) \neq 0$$

be an analytic function in the disc |u| < r, (r > 1) and

$$p_k(x) = \sum_{i=0}^{k} a_i \frac{x^{k-i}}{(k-i)!}, (k \in \mathbb{N})$$

be the Appell polynomials defined by the identity

$$g(u)e^{ux} = \sum_{k=0}^{\infty} p_k(x) u^k \qquad (1.2)$$

Ibrahim Bu¨yu¨kyazici et. al. [2] gave a chlodowsky type generalization of Jakimovski - Leviatan operators given by

$$P_n^*(f;x) = \frac{e^{-\frac{n}{b_n}x}}{g(1)} \sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right) f\left(\frac{k}{n}b_n\right) \qquad (1.3)$$

with bn a positive increasing sequence with the properties

$$\lim_{n\to\infty} b_n = \infty \quad and \quad \lim_{n\to\infty} \frac{b_n}{n} = 0 \qquad (1.4)$$

Motivated by the operators given in (1.2) and (1.3) we consider the following integral modification of the operator (1.3):

$$A_n(f;x) = \frac{n}{b_n} \frac{e^{-\frac{n}{b_n}x}}{g(1)} \sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right) \int_0^{\infty} \frac{e^{-\frac{n}{b_n}t}}{k!} \left(\frac{n}{b_n}t\right)^k f(t)dt \qquad (1.5)$$

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where bn is a positive increasing sequence satisfying (1.4).We see by construction of the operator An that the condition for positivity given by Wood for operator Pn is applicable here also. So throughout this paper we will assume that the operators An are positive.

Approximation properties of An(f ; x)

We denote by CE[0, ∞) the set of all continuous functions f on [0, ∞) with the property that |f (x)| ≤ βeαxfor all x ≥ 0 and some positive finite α and β.

Following lemma was given in [3].

Lemma 2.1. From (1.2) we have

$$\sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right) = g(1)e^{\frac{n}{b_n}x} = \phi_0(x)$$

$$\sum_{k=0}^{\infty} kp_k\left(\frac{n}{b_n}x\right) = \left(\frac{n}{b_n}xg(1) + g'(1)\right)e^{\frac{n}{b_n}x} = \phi_1(x)$$

,

$$\sum_{k=0}^{\infty} k^2p_k\left(\frac{n}{b_n}x\right) = \left(g(1)\frac{n^2}{b_n^2}x^2 + (g(1)+2g'(1))\frac{n}{b_n}x + g'(1) + g''(1)\right)e^{\frac{n}{b_n}x} = \phi_2(x)$$

$$\begin{aligned}\sum_{k=0}^{\infty} k^3p_k\left(\frac{n}{b_n}x\right) = \Big(&g(1)\frac{n^3}{b_n^3}x^3 + (4g(1)+3g'(1))\frac{n^2}{b_n^2}x^2 + (g(1)+8g'(1)+3g''(1))\frac{n}{b_n}x\\ &+ g'(1) + 4g''(1) + g'''(1)\Big)e^{\frac{n}{b_n}x} = \phi_3(x)\end{aligned}$$

$$\begin{aligned}\sum_{k=0}^{\infty} k^4p_k\left(\frac{n}{b_n}x\right) = \Big(&g(1)\frac{n^4}{b_n^4}x^4 + (10g(1)+4g'(1))\frac{n^3}{b_n^3}x^3 + (14g(1)+30g'(1)+6g''(1))\frac{n^2}{b_n^2}x^2\\ &+ (g(1)+28g'(1)+30g''(1)+4g'''(1))\frac{n}{b_n}x + g'(1) + 14g''(1) + 10g'''(1)\\ &+ g^{(4)}(1)\Big)e^{\frac{n}{b_n}x} = \phi_4(x)\end{aligned}$$

Using lemma (2.1) and eq. (1.5) we get following results.

Lemma 2.2. The operators An defined by eq. (1.5) satisfy,

$$A_n(e_0;x) = 1$$

$$A_n(e_1;x) = x + \frac{b_n}{n}\left(1 + \frac{g'(1)}{g(1)}\right)$$

$$A_n(e_2;x) = x^2 + \frac{b_n}{n}\frac{(4g(1)+2g'(1))}{g(1)}x + \left(\frac{b_n}{n}\right)^2\frac{(2g(1)+4g'(1)+g''(1))}{g(1)}$$

$$\begin{aligned}A_n(e_3;x) = x^3 &+ \left(\frac{b_n}{n}\right)\frac{(10g(1)+3g'(1))}{g(1)}x^2 + \left(\frac{b_n}{n}\right)^2\frac{(18g(1)+20g'(1)+3g''(1))}{g(1)}x\\ &+ \left(\frac{b_n}{n}\right)^3\frac{(6g(1)+18g'(1)+10g''(1)+g'''(1))}{g(1)}\end{aligned}$$

$$\begin{aligned}A_n(e_4;x) = x^4 &+ \left(\frac{b_n}{n}\right)\frac{(20g(1)+4g'(1))}{g(1)}x^3 + \left(\frac{b_n}{n}\right)^2\frac{(89g(1)+60g'(1)+6g''(1))}{g(1)}x^2\\ &+ \left(\frac{b_n}{n}\right)^3\frac{(96g(1)+178g'(1)+60g''(1)+4g'''(1))}{g(1)}x\\ &+ \left(\frac{b_n}{n}\right)^4\frac{(24g(1)+96g'(1)+89g''(1)+20g'''(1)+g^{(4)}(1))}{g(1)}\end{aligned}$$

where ei(t) = ti, i = 0, 1, 2, 3, 4.

Proof. From the definition of An(f ; x) and lemma (2.1) , we have

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$$A_n(e_0;x)=\frac{n}{b_n}\frac{e^{-\frac{n}{b_n}x}}{g(1)}\sum_{k=0}^{\infty}p_k\left(\frac{n}{b_n}x\right)\int_0^{\infty}\frac{e^{-\frac{n}{b_n}t}}{k!}\left(\frac{n}{b_n}t\right)^k dt$$

$$=\frac{e^{-\frac{n}{b_n}x}}{g(1)}\sum_{k=0}^{\infty}p_k\left(\frac{n}{b_n}x\right)\frac{\Gamma(k+1)}{k!}=\frac{e^{-\frac{n}{b_n}x}}{g(1)}\phi_0(x)=1$$

$$A_n(e_1;x)=\frac{n}{b_n}\frac{e^{-\frac{n}{b_n}x}}{g(1)}\sum_{k=0}^{\infty}p_k\left(\frac{n}{b_n}x\right)\int_0^{\infty}\frac{e^{-\frac{n}{b_n}t}}{k!}\left(\frac{n}{b_n}t\right)^k t\,dt$$

$$=\left(\frac{b_n}{n}\right)\frac{e^{-\frac{n}{b_n}x}}{g(1)}\sum_{k=0}^{\infty}p_k\left(\frac{n}{b_n}x\right)\frac{\Gamma(k+2)}{k!}=\left(\frac{b_n}{n}\right)\frac{e^{-\frac{n}{b_n}x}}{g(1)}(\phi_1(x)+\phi_0(x))$$

$$=x+\frac{b_n}{n}\left(1+\frac{g'(1)}{g(1)}\right)$$

$$A_n(e_2;x)=\frac{n}{b_n}\frac{e^{-\frac{n}{b_n}x}}{g(1)}\sum_{k=0}^{\infty}p_k\left(\frac{n}{b_n}x\right)\int_0^{\infty}\frac{e^{-\frac{n}{b_n}t}}{k!}\left(\frac{n}{b_n}t\right)^k t^2 dt$$

$$=\left(\frac{b_n}{n}\right)^2\frac{e^{-\frac{n}{b_n}x}}{g(1)}\sum_{k=0}^{\infty}p_k\left(\frac{n}{b_n}x\right)\frac{\Gamma(k+3)}{k!}$$

$$=\left(\frac{b_n}{n}\right)^2\frac{e^{-\frac{n}{b_n}x}}{g(1)}\sum_{k=0}^{\infty}p_k\left(\frac{n}{b_n}x\right)(k^2+3k+2)$$

$$=\left(\frac{b_n}{n}\right)^2\frac{e^{-\frac{n}{b_n}x}}{g(1)}(\phi_2(x)+3\phi_1(x)+2\phi_0(x))$$

$$=x^2+\frac{b_n}{n}\frac{(4g(1)+2g'(1))}{g(1)}x+\left(\frac{b_n}{n}\right)^2\frac{(2g(1)+4g'(1)+g''(1))}{g(1)}$$

For e3 we have

$$A_n(e_3;x)=\frac{n}{b_n}\frac{e^{-\frac{n}{b_n}x}}{g(1)}\sum_{k=0}^{\infty}p_k\left(\frac{n}{b_n}x\right)\int_0^{\infty}\frac{e^{-\frac{n}{b_n}t}}{k!}\left(\frac{n}{b_n}t\right)^k t^3 dt$$

$$=\left(\frac{b_n}{n}\right)^3\frac{e^{-\frac{n}{b_n}x}}{g(1)}\sum_{k=0}^{\infty}p_k\left(\frac{n}{b_n}x\right)\frac{\Gamma(k+4)}{k!}$$

$$=\left(\frac{b_n}{n}\right)^3\frac{e^{-\frac{n}{b_n}x}}{g(1)}\sum_{k=0}^{\infty}p_k\left(\frac{n}{b_n}x\right)(k^3+6k^2+11k+6)$$

$$=\left(\frac{b_n}{n}\right)^3\frac{e^{-\frac{n}{b_n}x}}{g(1)}(\phi_3(x)+6\phi_2(x)+11\phi_1(x)+6\phi_0(x))$$

$$=x^3+\left(\frac{b_n}{n}\right)\frac{(10g(1)+3g'(1))}{g(1)}x^2+\left(\frac{b_n}{n}\right)^2\frac{(18g(1)+20g'(1)+3g''(1))}{g(1)}x$$

$$+\left(\frac{b_n}{n}\right)^3\frac{(6g(1)+18g'(1)+10g''(1)+g'''(1))}{g(1)}$$

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and

$$A_n(e_4;x) = \frac{n}{b_n}\frac{e^{-\frac{n}{b_n}x}}{g(1)}\sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right)\int_0^{\infty}\frac{e^{-\frac{n}{b_n}t}}{k!}\left(\frac{n}{b_n}t\right)^k t^4 dt$$

$$= \left(\frac{b_n}{n}\right)^4\frac{e^{-\frac{n}{b_n}x}}{g(1)}\sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right)\frac{\Gamma(k+5)}{k!}$$

$$= \left(\frac{b_n}{n}\right)^4\frac{e^{-\frac{n}{b_n}x}}{g(1)}\sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right)(k^4+10k^3+35k^2+50k+24)$$

$$= \left(\frac{b_n}{n}\right)^4\frac{e^{-\frac{n}{b_n}x}}{g(1)}(\phi_4(x)+10\phi_3(x)+35\phi_2(x)+50\phi_1(x)+24\phi_0(x))$$

$$= x^4 + \left(\frac{b_n}{n}\right)\frac{(20g(1)+4g'(1))}{g(1)}x^3 + \left(\frac{b_n}{n}\right)^2\frac{(89g(1)+60g'(1)+6g''(1))}{g(1)}x^2$$

$$+ \left(\frac{b_n}{n}\right)^3\frac{(96g(1)+178g'(1)+60g''(1)+4g'''(1))}{g(1)}x$$

$$+ \left(\frac{b_n}{n}\right)^4\frac{(24g(1)+96g'(1)+89g''(1)+20g'''(1)+g^{(4)}(1))}{g(1)}$$

The following result can be easily derived from the previous lemma.

Lemma 2.3. The operators An defined by eq. (1.5) satisfy,

$$A_n(t-x;x) = \frac{b_n}{n}\left(1+\frac{g'(1)}{g(1)}\right) \tag{2.1}$$

$$A_n((t-x)^2;x) = 2x\frac{b_n}{n} + \left(\frac{b_n}{n}\right)^2\frac{(2g(1)+4g'(1)+g''(1))}{g(1)} \tag{2.2}$$

$$A_n((t-x)^4;x) = \left(\frac{b_n}{n}\right)^2\frac{(29g(1)+4g'(1))}{g(1)}x^2 + \left(\frac{b_n}{n}\right)^3\frac{(72g(1)+106g'(1)+20g''(1))}{g(1)}x$$

$$+ \left(\frac{b_n}{n}\right)^4\frac{(24g(1)+96g'(1)+89g''(1))+20g'''(1)+g^{(4)})}{g(1)} \tag{2.3}$$

Theorem 2.4. For f ∈ CE[0, ∞), the operators An converge uniformly to f on

[0, a], a > 0 as n ∈ N.

Proof. From lemma (2.2) we have

lim An(ei; x) = ei(x), i = 0, 1, 2.

n→∞

On applying Korovkin theorem [1] we get the desired result.

Order Of Approximation

In this section we give an estimate of the order of approximation of a function f ∈ CE[0, ∞) by means of the operator An , using the first order modulus of conti- nuity.

Let f ∈ C[0, b].The modulus of continuity of f denoted by ω(f, δ),is defined to be

$$\omega(f,\delta) = \sup_{|s-x|<\delta, s,x\in[0,b]} |f(s)-f(x)|$$

The modulus of continuity of the function f in C[0,b] gives the maximum oscillation of f in any interval of length not exceeding δ > 0.

It is well known that for any δ > 0 and each s ∈ [0, b]

$$|f(s)-f(x)| \leq \omega(f,\delta)\left(1+\frac{|s-x|}{\delta}\right) \qquad (3.1)$$

Theorem 3.1. Forf ∈ CE[0, ∞) and x ∈ [0, a], a > 0 we have

$$|A_n(f;x)-f(x)| \leq \left[1+\sqrt{2a+\frac{b_n}{n}\frac{(2g(1)+4g'(1)+g''(1))}{g(1)}}\right]$$

Proof. By using eq. (3.1) linearity of operators An we obtain

$$\begin{aligned}|A_n(f;x)-f(x)| &\leq A_n(|f(t)-f(x)|;x)\\ &\leq \omega(f,\delta)A_n\left(1+\frac{|t-x|}{\delta};x\right)\\ &= \omega(f,\delta)\left(1+\frac{1}{\delta}\left(\frac{n}{b_n}\frac{e^{-\frac{n}{b_n}x}}{g(1)}\sum_{k=0}^{\infty}p_k\left(\frac{n}{b_n}x\right)\int_0^\infty \frac{e^{-\frac{n}{b_n}t}}{k!}\left(\frac{n}{b_n}t\right)^k|t-x|dt\right)\right)\end{aligned}$$

Using the Cauchy - Schwarz inequality we obtain

$$|A_n(f;x)-f(x)|$$

$$\leq \omega(f,\delta)\left(1+\frac{1}{\delta}\left(\frac{n}{b_n}\frac{e^{-\frac{n}{b_n}x}}{g(1)}\sum_{k=0}^{\infty}p_k\left(\frac{n}{b_n}x\right)\int_0^\infty \frac{e^{-\frac{n}{b_n}t}}{k!}\left(\frac{n}{b_n}t\right)^k(t-x)^2dt\right)^{1/2}\right)$$

$$= \omega(f,\delta)\left(1+\frac{1}{\delta}\sqrt{A_n((t-x)^2;x)}\right) \qquad (3.2)$$

From eq. (2.2) ,for x ∈ [0, a], a > 0 we obtain

$$A_n((t-x)^2;x) \leq 2a\frac{b_n}{n}+\left(\frac{b_n}{n}\right)^2\frac{(2g(1)+4g'(1)+g''(1))}{g(1)} \qquad (3.3)$$

Taking $\delta = \sqrt{\frac{b_n}{n}}$ and using eq. (3.3) in eq. (3.2) we get the desired result.

4. Theorem of Voronovskaja - type

Now we give a Voronovskaja - type relation for the operator An.

Theorem 4.1. If f ∈ C2 [0, ∞) then

$$\lim_{n\to\infty}\frac{n}{b_n}\{A_n(f;x)-f(x)\} = f'(x)\left[1+\frac{g'(1)}{g(1)}\right]+xf''(x).$$

uniformly for x ∈ [0, a], a > 0.

Proof. For a fixed x0 ∈ [0, ∞),by Taylor's formula we have for every t ∈ [0, ∞)

$$f(t)=f(x_0)+f'(x_0)(t-x_0)+\frac{1}{2}f''(x_0)(t-x_0)^2+\psi(t;x_0)(t-x_0)^2, \qquad (4.1)$$

where ψ(t; x0) is a function belonging to the space CE[0, ∞) and limt→x0 ψ(t; x0) = 0.Then by (4.1) and lemma (2.2) we can write for every n ∈ N,

$$\begin{aligned}\frac{n}{b_n}[A_n(f;x_0)-f(x_0)] &= \frac{n}{b_n}f'(x_0)A_n(t-x_0;x_0)+\frac{1}{2}\frac{n}{b_n}f''(x_0)A_n((t-x_0)^2;x_0)\\ &+\frac{n}{b_n}A_n(\psi(t;x_0)(t-x_0)^2;x_0) \qquad (4.2)\end{aligned}$$

From eqs. (2.1) and (2.3) we have

$$\lim_{n\to\infty} \frac{n}{b_n} A_n(t-x_0;x_0) = \left(1+\frac{g'(1)}{g(1)}\right) \quad (4.3)$$

$$\lim_{n\to\infty} \frac{n}{b_n} A_n((t-x_0)^2;x_0) = 2x_0 \quad (4.4)$$

By Cauchy-Schwarz inequality we get for n ∈ N

$$\left|\frac{n}{b_n} A_n(\psi(t;x_0)(t-x_0)^2;x_0)\right| \le \{A_n(\psi^2(t;x_0);x_0)\}^{\frac{1}{2}} \left\{\left(\frac{n}{b_n}\right)^2 A_n((t-x_0)^4;x_0)\right\}^{\frac{1}{2}} \quad (4.5)$$

From (2.3) we have

$$\lim_{n\to\infty} \left(\frac{n}{b_n}\right)^2 A_n((t-x_0)^4;x_0) = \frac{(29g(1)+4g'(1))}{g(1)} x_0^2 \quad (4.6)$$

Let ϕ(t, x0) = ψ2(t; x0), t ≥ 0.Then ϕ(t, x0) ∈ CE[0, ∞) and $\lim_{t\to x_0} \phi(t,x_0) = 0$

Then from Theorem (2.4) we have

lim An(ψ2(t; x0); x0) = lim An(ϕ(t; x0); x0) = ϕ(x0; x0) = 0 (4.7)

n→∞ n→∞

uniformly with respect to x0 ∈ [0, a]. So by eqs. (4.6)-(4.7) we have

$$\lim_{n\to\infty} \frac{n}{b_n} A_n(\psi(t;x_0)(t-x_0)^2;x_0) = 0 \quad (4.8)$$

then, taking the limit as limn→∞ in (4.2) and using (4.2),(4.4) and (4.5) we have

$$\lim_{n\to\infty} \frac{n}{b_n}\{A_n(f;x)-f(x)\} = f'(x)\left[1+\frac{g'(1)}{g(1)}\right] + xf''(x).$$

Hence the theorem.

2. CONCLUSION

In the present paper we considered a Chlodovski type integral modification of Jakimovski- Leviatan operators involving Appell Polynomials . We proved that these new operators are approximation process. We have also given an asymptotic estimate through Voronovskaja - type result for these operators.

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