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EVERY PLANAR GRAPH WITHOUT ADJACENT TRIANGLES OR 7-CYCLES IS (3, 1)∗−CHOOSABLE

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ABSTRACT

In a graph G, a list assignment L is a function that it assigns a list $L(v)$ of colors to each vertex $v \in V(G)$. An $(L,d)^*$-coloring is a mapping β that assigns a color $\beta(v) \in L(v)$ to each vertex $v \in V(G)$ so that at most impropriety d neighbors of v are the same color with $\beta(v)$. A graph G is said to be $(k,d)^*$-choosable if it admits an $(L,d)^*$-coloring for every list assignment L with $|L(v)| \geq k$ for all $v \in V(G)$. In this paper, we prove that every planar graph with neither adjacent triangles nor 7 -cycles is $(3,1)^*$-choosable. In 2016, Min Chen, Andre Raspaud and Weifan Wang proved that every planar graph with neither adjacent triangles nor 6 -cycles is $(3,1)^*$-choosable.

Keywords: Planar graphs, improper choosability, cycle.

1. INTRODUCTION

A k-coloring of G is a mapping β from $V(G)$ to a color set $\{1,2,\cdots,k\}$ such that $\beta(x) \neq \beta(y)$ for any adjacent vertices x and y. A graph is $k-$ colorabe if it has a k-coloring. Cowen et al.(1986) considered defective coloring of graphs. A graph G is said to be d-improper k - colorable, or simply, $(k,d)^* -$ colorable, if the vertices of G can be colored with k colors in such a way that vertex has at most d neighbors receiving the same color as itself. Clearly, a $(k,0)^* -$ coloring is an ordinary proper k - coloring.

A list assignment of G is a function L that assigns a list $L(v)$ of col- or $\beta(v) \in L(v)$ to each vertex $v \in V(G)$ so that at most d neighbors of v receive color $\beta(v)$. A graph is k-choosable with impropriety of integer d, or simply $(k,d)^* -$ choosable, if there exists an $(L,d)^*$-coloring for every is just the ordinary k-choosability introduced by Erdős et al. (1979) and independently by Vizing (1976). A famous and classic result given by Thomassen (1994) is that every planar graph is $(5,0)^*$-choosable. However, Voigt (1993) showed that not all planar graphs are $(4,0)^*$-choosable by establishing a non- $(4,0)^*$-choosable planar graph.

In 1999, $\bar{S}$ rekovski(1999a) and Eaton and Hull (1999) independently introduced the concept of list improper coloring. They showed that planar graphs are $(3,2)^*$-choosable and outerplanar graphs are $(2,2)^*$-choosable. They are both improvement of the results shown in Cowen et al. (1986) which say that planar graphs are $(3,2)^*$-colorable and outerplanar graph$s$ are $(2,2)^*$ colorable. Note that there exist non- $(2,2)^*$-colorable planar graphs and non- $(2,1)^*$-colorable outerplanar graphs which were constructed in Cowen et al (1986). Let $g(G)$ denote the girth of a graph G, i.e., the length of a shortest cycle in G. The $(k,d)^*$-choosability of planar graph G with given $g(G)$ has been investigated by $\overline{\text{Srekovski}}$ (2000). He proved that every planar graph G is $(2,1)^*$-choosable if $g(G) \geq 9, (2,2)^*$-choosable if $g(G) \geq 7, (2,3)^*$-choosable if $g(G) \geq 6$, and $(2,d)^*$-choosable if $d \geq 4$ and $g(G) \geq 5$. The first two results were strengthened by Havet and Sereni (2006) who proved that every planar graph G is $(2,1)^*$-choosable if $g(G) \geq 8$ and $(2,2)^*$-choosable if $g(G) \geq 6$. Recently, Cushing and Kierstesad (2010) proved that every planar graph is $(4,1)^*$-choosable. So it would be interesting to investigate the sufficient conditions of $(3,1)^*$-choosability of subfamilies of planar graphs where some families of cycles are forbidden. Slrekowski prowed in Srekovski (1999b) that every planar graph without 3 -cycles is $(3,1)^*$-choosable. Lih et al.(2001) proved that planar graphs without 4 - and l-cycles are $(3,1)^*$-choosable, where $l \in \{5,6,7\}$. Later, Dong and Xu (2009) proved that planar graphs without 4- and l-cycles are $(3,1)^*$-choosable, where $l \in \{8,9\}$. These two results were improved further by Wang and Xu(2013) who showed that every planar graph without 4 -cycles is $(3,1)^*$-choosable. More recently, Chen and Raspaud (2014) proved that every planar with neither adjacent 4 -cycles nor 4 -cycles adjacent to 3-cycles is $(3,1)^*$-choosable. This absorbs above results in Lih et al. (2001), Dong and Xu (2009), Wang and Xu (2013). Then, Min Chen, Andre Raspaud and Weifan Wang (2016) prowed that every planar graph with neither adjacent triangles nor 6 -cycles is $(3,1)^*$-choosable.

Theorem 1.1 Every planar gruph with neither adjacent triangles nor 7. cycles is $(3,1)^*$-choosable.

The proof of Theorem 1.1 is done in the section 3.

2 Notation

All graphs considered in this paper are finite, simple and undirected without multiple exges. Call a graph G planar if it can be embedded into the plane so that its edges meet only at their ends. Any such particular embedding of a planar

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graph is called a plane graph. For a plane graph G, we tise V, E, F, Δ and $\delta(V(G), E(G), F(G), \Delta(G), \delta(G))$ to denote its vertex set, edge set, face set, maximum degree and minimum degree, respectively. For a vertex $v \in V$, the degree of v in G, denoted by $d_G(v)$, or simply $d(v)$, is the number of edges incident with v in $G. |V(G)|$ and $|E(G)|$ are order and size. The neighborhood of v in G, denoted by $N_G(v)$, or simply $N(v)$, consists of all vertices adjacent to v in G. Call v a k-vertex, or a k^+-vertex, or a k^--vertex if $d(v) = k$, or $d(v) \geq k$, or $d(v) \leq k$, respectively. A similar notation will be used for cycles and faces. For a face $f \in F$,

the number of edges of the boundary of f (where cut edge, if any, is counted twice), denoted by $d(f)$, is called the degree of f. Analogously, the notations above for vertices will be applied to faces. We write $f = [v_1 v_2 \cdots v_k \mid$ if $v_1, v_2, \cdots, v_k$ are consecutive vertices on f in a cyclic order, and say that f is a $(d(v_1), d(v_2), \cdots, d(v_k))$-face. Next, let f_i be the face with vv_i and vv_{i+1} as two boundary edges for $i = 1,2, \cdots, d(v)$, where indices are taken modulo $d(v)$ and define $d(v) + 1 = 1$. Let v be a vertex, and v is a $3 - vertex$ in G such that the three neighbors vertices adjacent with v. An edge xy is called a $(d(x), d(y))$-edge, and x is called a $d(x)$-neighbor of y. A $k -$ cycle is a cycle of length k. In this paper, a 3 -face is often called a triangle. Call a vertex or an edge triangular if it is incident with a triangle. Otherwise, a vertex or an edge iso-triangular if it is not incident with a triangle but its neighbor vertex is incident with triangle. Then 4-face is often called a quadrilateral. Two cycles or two faces are intersecting if they have at least one vertex in common; and are adjacent if they have at least one edge in common. Again, 4-face is called a quadrilateral in which two triangles are adjacent.

We define the following notation:

- Let u be a 4 -vertex. If u is incident with f_1, f_2, f_3 and f_4 so that $f_1 = |uu_1u_2| = (3,4,5^+)$-face and then $d(f_3) = 4$ and $d(f_2) = d(f_4) = 8^+ -$face. It is called a 4-light vertex. Shown in Figure 1.

  $8+ -$ face 8+-face

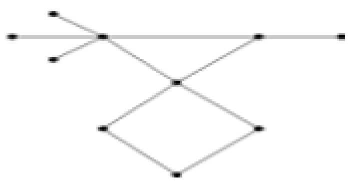


Figure 1:

Definition 2.1 Let f be 3 -face such that $f = [u\psi_1 u_2]$ and ef be an cdge incident with f. i.e., $e_{\mathrm{um}_1}, \epsilon_{\mathrm{vu}_2}, e_{\mathrm{u}_1 w_1}$ can be written by e_f.

Definition 2.2 - A s-verter is said to be poor if it is incident with one 3-face and two 4 -faces. Then it is colled 3 -poor.

- Let u be a 4 -vertex and $f = [\psi u_1 u_2 \mid$ be a 9 -face. If u is incident with one 3-fare, one 4-face and one 5-face adjacrnt with ef and another is 6 -face, then it is said to be 4 -poor. (OR)
- A 4 -vertex is said to be poor if it is incident with one 3 -face and tuo of e_f incident with one 4 -face and one 5 -face and another is 6 -face. Then it is called 4-poor.
- Let u be a 5-vertex and $f = [\pi u_1 u_2 \mid$ be a 9-face. If u is incident with one 3 -face and both one 4 -face and one 5 -face aljacent with e_f and others' two are $6^+ - \int ace$ and $5^+ - \int ace$, then it is said to be $5 - poor$. (OR)

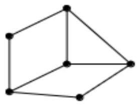
  A 5-vertex is said to be poor if it is incident with one 3-face and tuo of ef incident with one 4-face and one 5-face and others are incident with 6^+-face and 5^+-face. Then it is called 5 -poor.

Definition 2.3 - A 3 -vertex is suid to be semi-poor if it is incident with three 4 -faces. Then it is called 3 -semi-poor.

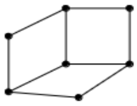
- A 4-vertex is said to be semi-poor if it is incident with one 3 -face adjacent to one 4-face and one 4-face adjacent to one $-face. Then it is also called a semi-poor-I verter.
- A 4-vertex is said to be semi-poor if it is incident with one 3 -face adjacent to one 4-face and one 4-face adjacent to one 4-face. Then it is also called a semi-poor-II vertex.

- A f-vertex is said to be semi-poor if it is incident with one 3 -face adjacent to one 5-face and one f-face adjacent to one 9 -face. Then it is also called a semi-poor-III vertex.
- A 4-verter is said to be semi-poor if it is incident with one 3-face adjacent to one 5-face and one 4-face adjacent to one 4-face. Then it is also called a semi-poor-IV vertex.

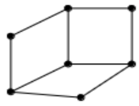
  Definition 2.4 - A S-vertex is said to be full-poor if it is incident with one 3 -face, one 5 -face and 8^{+}-face. Then it is culled 3 -full-poor.
- A 4-verter is said to be full-poor if it is incident with one 4 -face adjacent to one 3-face and one 4-face adjacent to one 3-face. Then it is also called a full-poor-I vertex.
- A f-vertex is said to be full-poor if it is incident with one f-face adjacent to one 3 -face and one 4-face adjacent to one 4-face. Then it is also called a full-poor-II vertex.
- A f-verter is said to be full-poor if it is incident with one 4 -face adjacent to one 4-face and one 4-face adjacent to one 4 -face. Then it is also called a full-poor-III verter.



3 -poor



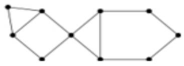
3 Semi -poor



3 -Full poor

Figure: 2

4-poor



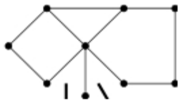
4-semi-poor I

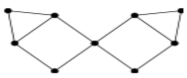
4-semi-poor II

Figure: 3

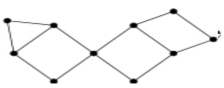
4- Semi Poor III

4 -Semi Poor IV

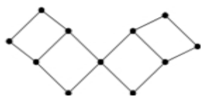




4- full Poor 1

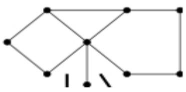


4- full Poor II



4- full Poor III

Figure 5:



6+-Face 5+-face

5 Poor

Theorem 2.5 (Chen [1]). Every planar graph neither adjacent triangle nor 6 cycle is $(3,1)^{*}$-choosable.

Theorem 2.6 (Chen [2]). Every planar gruph without &-cycles adjacent to 3. and 4-cycles is $(3,1)^{+}$-choosable.

Lemma 2.7 (Lih, Wang, Zhang [9]).
(A1) $\delta(G) \geq 3$.
(A 2) No two adjacent s-vertices.

Lemma 2.8 Let f be (3,4,5)-face. Then all vertices of f are poor.
Proof: Let $f=[xyz]=(3,4,5)$-face and then $x_1 \in N(x), y_1, y_2 \in N(y)$ and $z_1, z_2, z_3 \in N(z)$. Suppose to the contrary that there is no poor vertex of f in G. Let $G'=\{x, y, z, x_1, y_1, y_2, z_1, z_2, z_3\}$. By minimality of G, suppose that $G-G'$ has an $(L, 1)^{*}$-coloring of β.

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First, for $d(x)=3$, without loss of generality, let xx_1y_1y be a quadrilateral and $e_{x=}$ be not incident with 4 -face. We may prowide the colors $\beta(y)=\beta(x_1)=\beta(z_1)=1$ and $\beta(y_1)=\beta(z)=2$. We must have the color $\beta(x)$ with $L(x)\setminus\{\beta(y)\cup\beta(z)\cup\beta(x_1)\}$. So, we choose the color $\beta(x)$ with 3. If we recolor $\beta(x_1)$ with $L(x_1)\setminus\{\beta(y_1)\cup\beta(x_1')\}$, then we will get the color of the same $\beta(x)$. If we recolor $\beta(x_1)$ with 3 , we can exchange the colors $\beta(x)$ and $\beta(z)$. However, since e_{xx} is not incident with 4 -face, it means that it is incident with 8 -face. So, y_1 and x_1' can be adjacent to each other. If $y_1x_1x_1'$ is a triangle, we must have the color $\beta(x_1')$ with 3 . So, it is impossible for the color $\beta(x_1)$ with 3 . If $y_1x_1x_1'$ is not a triangle, y_1y_2 can be a triangle. So, we can assume that the colors $\beta(x_1)$ and $\beta(y_2)$ with 3 . Since e_{xz} is not incident with 4 -face, so $x_1'\neq z_1$. So, we could have the colors $\beta(x_1')$ and $\beta(z_1)$ are the same. Then we change the colors $\beta(z)$ and $\beta(z_1)$. It is contradiction for x vertex.

Secondly; for $d(y)=4$ and $d(z)=5$, we have proved that x is a poor vertex. Without loss of generality, we have x_1xyy_1 and x_1xzz_1 are quadrilaterals and then we cannot have both yy_1y_2 is a triangle and yy_1*y_2 is a quadrilateral. So, we may assume that zz_2z_3 is a triangle. Since e_{yz} is not incident with $4-,5-,6$-faces. Without loss of generality, let $L(x)=\ L(y_1)=L(y_2)=L(z_1)=\{1,2,3\}, L(y)=L(z_2)=\{1,2,4\}, L(z)=\ L(x_1)=\{1,3,4\}$ and $L(z_3)=\{2,3,4\}$. If we provide the colors $\beta(y_1)=\beta(y_2)=\beta(z_2)=1, \beta(z_1)=3$ and $\beta(y)=\beta(z_3)=2$, then we must have

the colors $\beta(x_1)$ with 4 and $\beta(z)$ with 4 . We can give the color $\beta(x)$ with $L(x)\setminus\{a(n)\cup^{\beta}(s)\cup^{3(}(x),\}$ - If we recolor $d(\pi)$ with 4 , we matst eschustip: the culces if sal and 3(=). Howver, 2 & $L(s)$. It be impocosile. Thass, it is coutradiction iry assumption. Tharebser, the peool is eomplete.

Lemma 2.9 If / te at (4,4,4,4) fanc, then evry nerier of 4-fans atn be e 4-fight nerier. that x_i, W_nx_i asd w_j ate the neighbors of x,v,s,v, compesing of a tristugle with their usighloor whute $f\in\{1,2\}$. Suppoee to the coutrary that wune of $x,y,z,w=bi=4$-light wertex such that $d(\mathcal{A}_i)\geq 4$. where $A_i=\{x_i+x_i,z_i,w_i\}$. $i=\{1,2\}$. Let $G'=\{x,y,z,w,x_i,w_i,x_i,w_i\}, i=\{1,2\}$. By the minimality of C. $G-C'$ wilnibs an (L. 1)-cobsting of β. We will ecestille two casas.

Case (i) We may give colors with $\beta(x)$ and $\beta(x)$ ase the satne atal $\beta(y)$ und $\beta(x)$ abe also. So, let $\beta(x)=\beta(z)-1$ sad $\beta(y)=\beta(w)=2$. Tlus, we can dethuce that $\beta(a_i)\in\{2,3\}$ atul $a(b_i)\in\{1,3\}$, whese $a_i=\{x_i,z_i\}$ and $b_1=\{w,w\},, i\in\{1,2\}$. We coesbber three subt-cicass in the following-

Sub-case (i) Firsaly, fur x we will coutwillet x_1 sasd x_2 hawe to be incident with only case triaugle. By asentuption, we have $(x_1x_2x)=(3,4,4)$-lacte. We must have the cilors $\{\theta(x_2'),g(x_2'),\beta(x_2'')\}\subseteq\{1,2,3\}$. If $x_1x_1'x_2x_2$ is a quaulrilatital, we counot give the asmat ecobes $\beta(x_1),\beta(x_2')$ sad $\beta(x_2), So$, w may tosatmat that $\beta(x)=\beta(x_2)=1, \beta(x_2)=\beta(x_2)=2, \beta(x_2)=\beta(x_1)=3$. $\beta(x_1)=\beta(x_1)-\beta(x_2)=1$. Here, we must have the coloes $\beta(x_2)-2$. Ir we exriange the cobses $\beta(x_2)$ and $i(x_2)$, we mut trodor $\beta(x)$ with 2 or a. Mloriower, secobully, for the writex y, we will coctodiler in und yz have to be incibent with only one triauge. We may aseume that $\beta(y_1)=1, \beta(y_2)=3$. If $my_1y_2v_2$ is a qualtilateral, we have differat colors betworn y_2 und y_2. So. if we asoume that $\beta(y_2')=\beta(y_2)=2$, we mast have the colots $\beta(m_1')$ with 3. CBearly, we hume $M(y_2)-1$ acs $\beta(y_2)-3$. If we exclange the cobors $B(y_2)$

Sub-case (ii) Fur the vethex x_1, we will cousilie x_1 wad x_2 hune to be iracithet with trinagle We mase hane the colun $\{\beta(x_1),\vec{\theta}(x_2)\beta(x_2)\}\subseteq(1,2,3\}$. Lat $x_2x_2'x_2'$ be an traugle and $x_1x_1'x_2x_2$ be a quadrilateral. We may assume that $\beta(x_2)=2, \beta(x_2)=3, \beta(x_1')=A(x_2')=1$. Here, we muse lume the color $\beta(x_2')=2$. If we esclaage the colots $\beta(x_1)$ and AN $'_1$), iod thes the tivers $\beta(x_2)$ and $\beta(x_1')$, we mad recilor $j(x)$ with 3. Motowver, for the vertex y. we erill cusbser in and yz howe to be lircident with triangle. Lat yrrive be $\beta(y_2)=3. \beta(y_1')=\beta(y_2')=2.5a_1$, we tuit hane the cabe $\beta(y_2')=1$. If we exuthuge the cobors $A(x)$ and $\beta(y)$, it is impossible for $\beta(y_i)\leq(1,3)$. Thuse we will excthange the colors $\beta(y)$ sad $3(y_2)$. It is eoutralsetson by wevuruptson.

quadrilatizal. Let $\beta(x_1)=\beta(x_2')=2$ und $a(x_1')=3$. We must have the colors $\parallel(x_2)$ with 3 sad $\beta(x_2'')$ with 1 . Similarly, we will coteviller the wetes g. Lut $\beta(y_1)=\#(p_2')=1$ sad $\beta(y_1')=2$. We must obtain the colors $B(y_1)$ and w, where $i\in\{1,2\}$, wre incidont with ouly 8^+-fwee, uty zavighoe of x_1', prowe anly two vetios x and ψ.

Cher(ii) We may give colors vith $\beta(x)$ and $\beta(y)$ are dilleretat. So, let $\beta(x)=1$ asd $a(z)-2$ sad $\beta(y)-3$ and $\beta(w)=s$. We mutat lave the colkers $\beta(x_i)\in\{2,3\}, \beta(w)$€$\{1,2\}$, und $3(z_i)\in\{1,3\}$. whuse $i\in\{1,2\}$. Suppese that $a-3$. We mont have $\beta(m_i)\in\{1,2\}$. If we torthatge the culots $\beta(x)$ and $\beta(x_1)$, we most have colors $\theta(x)\in\{2,3\}$. If we huwe the colors $\beta(x)$ with 3 , it is imposible becanse of $\beta(y)=3$. So, these ba the colur $\beta(x)$ with 2 . If we earthange the colors $\beta(y)$ wad $\beta(y_1)$, wv unat hwve caloes $\beta(g)\in\{1,2\}$. If w h hume a colker if(s) with 2 , it is imposible. Sa. Here mast tee the toblot $\beta(y)$ with 1. Ur we exchutuget the edurs $\beta(z)$ and $\beta(z1)$, we must have mikers the cobots $\beta(w)$ with $R(w)\setminus\{\beta(w_i)\cup\beta(x)\cup\beta(z)\}$. Thus, it is ostutrwlietion Lise stuggostion.

Similarly, Fer the vertex $\pm$ and w, we can boluce that the resulting coloring is an $(L,1)^*$-coloring, which is a motraliction. Thutusere, the prout is ecmulete.

Lemmia 2.10 Lat f be a s-fere by $(3,4.4^+)$-feore.

(i) If 3-neriex is a 3-pour verter, then nave of tue f-wcrtions in a f-semipoor verter.

(ii) $J/$ a 8 -vertex in a S-poor verter, three the neighbors of the third writex not on ε_f is 4^+-twricis.

(iii) If e &-tertex is e I-poor wrier, then at wool dove tvertex of the neiyhlors of ture 4 -nerfices in 3 -verter. Proof: Lat $f=[xw_1w_2]=(3,4,4^+)$-fare and $N(x)=\{w_2,m_2,w_3\}$ and $N(u_i)=\{w_i+u_i\}$ where $i=\{1,2\}$.

Wer will prowe the lisst (i). Let u be a I-poour vertect. Suppocet to the coutrary that u_i is a 4 -momi-poor vertex in whinh $i=\{1,2\}$. We rose that α_i his a 4-vester incibent v_i' sud $v_{i=}'$ romd then u_i'' be inribent with v_{a}. Lat hes in $(L,1)^+$-coloring of A. Withont lass of giverality, let $a(x)=\Delta(x_2^*)=\beta(v_1^*)-1, \beta(u_1)-\beta(w_2')=2$ and $\beta(x_2)=\beta(v_1')-3$. Sinrs $|L|v_a)$ $|\geq 1$, si) we can cowiga the colce $\beta(\boldsymbol{u}')$ with 2 or 3 . If we ticolor $\beta(x)$ with 2 . then we must sosign the colot $\beta(w_1)$ with 1. But $\beta(v_1^*)=1$. Sa, we must be s quoulrilateral. So, $\beta(+)$ mimat be 2 . Hemee we must asaign the coler $\beta(v_1^*)$ with 3 . If we choose the coloss $\beta\left(u_j^c\right)$ with 3 und $\Rightarrow(x_1)$ with 2 , we tunst sosign the oblor $\beta(\mathbf{u}_1')$ with 2 .

If we clasuet the colors $\beta(\mathrm{n}_1^*)$ with 2 and with a, than we most assign the color $\beta(u_1)$ with 2 of 1 . If we doocedt $A(w_1)$ und $\beta(u_2^*)$ with 2 or a. If we choceet the colint $\beta(u_1')$ with 3 , than we mont we chocose the cober $\beta(v_1^r)$ with 1, then we most welign the cobors $\beta(v_1'')$ with 3 und $\theta(u_1)$ with 2 . If we dhowse that colors $\beta(v_2'')$ with 3 atad $\beta(i_2')$ with 3. then it is ootrialsetson loy mosumption. If we choose the cobse iM w_2) with 2 und $1(\dot{M}_2')$ with 3, then it is contmalintion. 4-[arso. Thuse, we have to kurw that it cuald be incidoul with 6^+ $-$farse. So. $d\left(u_j'\right)\geq 4$ und $d(v_1^n)=d(w_2^*)=3$. Horwever, w_1^r dad w_2^n catunt be haljacout to 3-virtex becsuse of w_1 and u_2 ase moe 4 -poour vertiose. Thasefore, the prout is coruplete. then nove of 4-fare ricidind with it rus be atjocnt to

(i) e 4-puor wortict.

(ii) a f-semi poive I terlicx and

(iii) a f-ncwai poost III twricr. incsbent with 4-poor verter.

Firstly, we will prove a 4-poce vertect incirleat with f_1-f_2 aul fa. Withonat bose of gowirulit, suppose that all of f_2-f_2 adil $\int_1$ ate incibont with a 4-poor vertex. Here, obvicrsly we will woontme that By minimnlity of G, suppose that $G-C'$ luct an $(L,1)^+$-ondoring of 3 . We wall cotsaider two civers.

Choe (i). We mov asoture that $\beta(v_1), \beta(u_2)$ und $a(x_2)$ ure the sume calors and $\beta(x), \beta(y)$ and $B(z)$ ure the sistae. So, we mary asodgn the colorn $\theta(\mathrm{m}_1)$. $\beta(v_2)$ sad $\beta(m_1)$ with 1 and thara the olies $\beta(x), \beta(y)$ und $\beta(z)$ with 2. Were, we must awiga the olor $\beta(u)$ with $L(x)\setminus\{\beta(u_1), \beta(x_2), 3(u_3)\}$ and we must sosign that cobor $\exists(a_l)$ with 3. Evibonty, 5-foer in 3-ecibring and fi-fare is 2 -eobsring. So, we mut whign the colors $\beta|a_2|$ wirh 1. Hete we will sosign that colle $\beta(u)$ with 3 . Here, we mist hawe ull eabes $a(x), \beta(y)$ adal $\beta(=)$ with 2 . If we esoluange the cobors $\beta(x)$ sall $\beta(u_L)$, we mat with $L(x_1)\setminus\{a_1'\}\}$. Sutace $\beta(x_2)=1$, it must be $\beta(x_1')=1$. Nirw, we cau have the cobor $\beta(x_2)$ wilh 2. It is contrulieticm. Motowne, since u_2 wal u_3 $\beta(\mathbf{x}_3)$ with 3. It bo comtruliction.

Further mure, since $|L(u)|=3$, we mod asoiga the culor $|x(x)$ with 2. $I(u_2)\rangle\{\beta(u_2')\}$ aul $\beta(v_1)$ will $I(u_3)\setminus\left(\beta(u_1)\right)$. Sos we mod have the colors Howerer, it is tamtratiction by asoumpticin.

Case (ii). We may wormat that $\beta|\mathrm{m}_1|, \beta|\mathrm{m}_2|$ sal $\beta|\mathrm{m}_3|$ are diffrimat. Evilıutlv. we mast have the colors $\beta(x), \beta|v|$ mal $\beta(z)$ are dillerent. We may cos ume that the colurs $\beta(u_3)$ with $1, M(m_2)$ with 2 wall $\beta(u_3)$ with 3 . So. we munt have the caloes $\beta(x)$ with $3. \beta(v)$ with 1 and $\beta(z)$ with 2 unt then ootulimasuly we must have the scibss $\beta(x))$ wilh $2, \beta(y)$ with 3 and \$$(z_1)$ with 1. If we asoiga the oolut $\beta(u)$ with 1 , than we mast necbor $\beta(u_1)$ with Hors, it in coulauliction.

If w swiga the cobst $\beta(u)$ with 2 , then we must becalor $\beta(w_2)$ with Howowt. it is botat rauliction. If we howign tlae colos $\beta(v)$ with 3,1 barn we with distirat $\beta(w_3)$. Howevet, it iev countrulictiom. obtiditiom (i). Thavelere, the proot is complete.

Corollary 2.12 Sappose to v is a 8-skmi-poior verfex in ataich $f_1=|v_1+\mathrm{I}_2|$. semi-poar tertions, Whre tbe there nertios of $\mathrm{y}_2, \mathrm{P}_2$ end is ark a^+-tertion.

(i) the threx noiglbors of = are 4^+ $-$mertios (i.e.. $f(N(x))\geq 4$) and

(ii) erarlly the werlex ty in either e f-poor werkex or a 5-poour nerfer.

Definition 2.14 (i) A vertex x in a $W(x)$-verficr inctilcut with at wobl u-trimigles and others are any foves. Ns merfer in callnd The -verter. Hore, ǀ T^{ra} | - the namber of w-triangles focidend wikh a neriex

(ii) A merter n is $d(n)$-merfer with dif $(u) \geq 4$ m miim n is inridenf will: craclly $\left\lfloor\frac{\mathrm{Ag}}{2}\right\rfloor$ 3-faos end exnctiy $\left\lfloor\frac{1\mathrm{f}}{2}\right\rfloor$ f-fores. It is said to be e micilont betwoce turo 3-fares.

Lemmin 2.15 Lal u be $T^{N|\infty)}$ - vertex iv C.Cubfilima:

3-facs, one 4-face and oue $8^{+}-$faca. n is coliul a special T^{3}-vertex Thes followitiog conditions: Lat u be $T^{-1/*)}-$ verter in G wilh $d(u) \geq 4$. 3-farrs, one 4 -farx and ane $8^{+}-$farr. tav S-ferses, wor 4 -fore, and then athers ere $\mathrm{g}^{+}-$farres lent with at most tiro 5^{+}-farcs and athers are incidfot writh at mool $\left\lceil\frac{d(a)-1}{-}\right\rceil - 18^{+}$-fores. Writh at mast $\left\lceil\frac{\mathbb{N}'}{4}\right\rceil 8^{+}-$faors

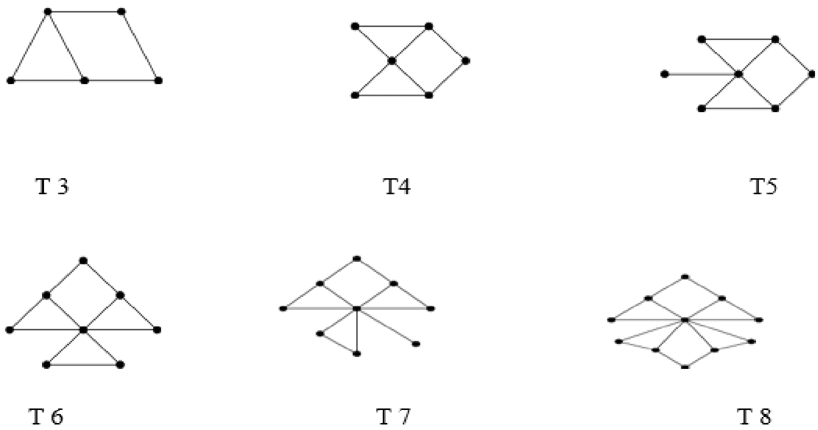

Figure: 7

Figure 7: in which there are incifent wilh af mast $\left\lfloor\frac{4\ \mathrm{w}}{2}\right\rfloor$ S.fares and et mowt $\left\lfloor\frac{4\ \mathrm{a}}{2}\right\rfloor$ 4-farrs, then there are at miast fao 5^{+}-fooss and $\left(\frac{df}{4}-\frac{1}{4}\right) 8^{+}-$faose

Corollary 2.17 J/u is a $T^{d/*)}-$ verfex $\langle d(u) \geq 9, d(u)-4n+5, n=1.2, \ldots)$ 4-faras, then there are at mast fimo 5^{+}-feors and $\left(\frac{4+\mathrm{f}}{2}-4\right) \mathrm{s}^{+}$-farse

2. DISCHARGING PROCESS

We soov upply a diathrging peocodure to mact a costrwlistson. We first difias the initial duarge furaction do on the vertions aral fices of G lyy let tings, $ch(v)=\Delta(v)-2b$ if $v \in V(G)$ und $ch(f)=(b-a)d(f)-2b, f \in F(G)$. We nute $a-\frac{3}{2}$ und $b=\frac{7}{2}$ ios that we get the initial function $cb(v)=\frac{3}{2}d(v)-7$ if $=\in V(G)^{2}$ and $da(f)^{2}-2\mathrm{df}(f)-7, f \in F(G)$. It followx from Ealer's formula $|V(G)|-|E(G)|+|F|G)$ ǀ -2 und the relatson

$$\sum_{v=V(D)} d(v)=\sum_{f\in F(G)} d(f)=2|E|(G)\,|$$

so) that the total sum of initial furction of the wrticis and fucos is equal to

$$\begin{aligned}\sum_{v\in V(G)} h(v)+\sum_{f\in F(G)} ch(f) &= \sum_{n=V(G)}\left(\frac{3}{2}d(v)-7\right)+\sum_{S\in F(G)}(2d(f)-7)\\ &=\frac{3}{2}[2|E(G)|)-7|V(G)|+2|2|E\Big|\ G\Big)\,|\Big)-7|F(G)|\\ &=-7(|V(G)|+|F(G)|-|E(G)|)=-14\end{aligned}$$

Since any diechurging proosolure preserves the total charge of C. if we can inflise suitahlie discharging rules to clange the initial churge funaction ah to the final charge function of on $V \cup F$ solh that $cM(x) \geq 0$ for all $x \in V \cup F$. thin

$$0 \leq \sum_{x \in VuF} \hat{d}'(x) = \sum_{x \in VL} dh(x) = -14,$$

a coutrudiction comipletling the proof of Thoverm 1.1 when G is 2-owntocted. Prool of Theorem 1.1 3, the following Lamua bi olwiuts.

Lemma 3.1 (i) In G, there is no adjwornt 3-farces.

(ii) In C, thers is e f-fare efjuront to at muat ture i-faris. Morwoser, whes e f-foor is erjarrat to ef frest ows 3-fars, the f-fare can be adjerant fo soe f-faxe eaript v is a S-poor worlex.

(iii) In G, there is a f-farx edjamout to al thest wor f-fers.

(iv) In G. there is a a-fers arfjarant to et moot ane 3-farr end no affarant to ars 4 -foces.

(v) In G, there io no 6-fear adjarant to e S-foos.

We will intecoluce the discharging rulis:

R 1. Chatge from a 4^{+}-face f

R. 1.1. If $\mathbb{d}(\Omega) = 4$, then f somal t to ewh incialeat vatex.

R. 1.2. If $A(f) = 5$, then f sombe to to ewh incidinat wrtex.

R. 1.3. If $A(f) = 6$, then f suake fof to ewch incideat votwx.

R 2.1. Suppoise to v is a 4 -liglat verter.

Let $f - | v_1 r_2 v) = (5^{+}, 3{,}4)$-five. Then v gets f from each from 8^{+}- lhee and $\frac{5}{7}$ from f. Afer that or gots $\frac{9}{-}$ Irom 8^{+}-fare and suali $\frac{13}{th}$ to f.

R 3. Suppoest to v be a poor vettox in which $f - [r_1 v_2 v_1]$ with $(v_1) \leq \mathrm{d}(\mathrm{v}_2) \leq \mathrm{d}(\mathrm{v}_2)$.

R. 3.1. Lat $d(v_2) = 3$ and v_2 le a 3 -post wortex. Thas v_1 gets $\frac{1}{2}$ Irom each 4 -face and f amale $\frac{1}{2}$ to y_1 –

R. 3.2. Lat $d(v_2) = 4$ anul r_2 be an 4-pose vertex. v_2 gds If from 5-frote aul from 6 -fuce and f gers $\frac{f}{f}$ from v_2.

R. 3.3. Let $d(v_1) = 5$ und v_2 le a 5-poot vertex. vy geb 3 frum 5 -fuct: ftrum 6^{+} $-$Lire und from 5^{+}-face amd then f grts ff from 19.

R 4. Suppoced to v be a 3-armi-poor vortex in which $f_1 - [v v_2 x_2], f_2 = |$ tr $_{2y} v_2$ | wad $f_a = |\mathrm{tr}_j * v_1|$ with f(v) d($\mathrm{v_i}$) whate $i \in \{1{,}2{,}3\}$.

Rt 4.1. Let $d(z) - 3$ and = be a 3-ami-poor vertex. Then v ges ff frotu each 4 -fince

R 4.2. Let $d(x) = d(\pi) = d(z) = 3$ ant thery be 3-somi-poos vetticts. as **r**.

R 5. Suppoceet to v_1 be a 3-full-poot vertex in which $f = |\mathrm{r}_1 v_2 v_3|$ with $d(v_1) \leq d(v_2) \leq d(v_2)$. Then v_1 gis 3 from 5-fuce and $\frac{18}{2}$ from 8^{+}-fact

R 6. Suppoceat to v be a 4-sumi-poor vertica in which $f_1 = |-v_1 v_2|, f_2 =$ from 8^{+} $-$fioe und it sumb $\frac{1}{2}$ to f_1.

R E.1.1 For $W_{(}(v_1) = d(v_2) = 3, v_2$ gets + from f_1 + from 4-fioce 8^{+} $-$face

R. 6.2 Lat v lee a 4 -ami-poor II vertix. Then = guts $\frac{1}{4}$ from f a wall $\frac{7}{3}$ Irotu 8^{+} $-$fhoce ind it sombe $\frac{1}{2}$ to f_1. Frum, 8^{+} $-$fact.

R 6.2 .2 For $d(r_4) = 3$, if the ontes neiglubor of v_1 is 4 -semifrom 8^{+}-lace. If the outer neighbor of r_2 is not 4 -wemi-poser vertex, then vy gets $\frac{2}{4}$ from fa und $\frac{1}{4}$ from 4 -lace wad $\frac{9}{-}$ from 8^{+} $-$fact

6.3 Lut v be a 4 -semi-poor III wirtex. Then v gets ff from f_1 ant from s^{+} $-$cars sud it amals if ta fi-

R. 6.3.1 For $d(v_1) = d(v_2) - 3, v_2$ gits $\frac{7}{-}$ from $fi, \frac{3}{5}$ from 5-fack sual $\frac{1}{-}$ from 8^{+} $-$bare whd thesi ve get $\frac{2}{3}$ from fa aul $\frac{9}{-}$ Ifom s^{+} $-$fact

6.4 Lat v be a 4-semi-poor IV vertex. Then v gots ff from fa aul Frum 8^{+} $-$tare and it somb af to f_1 frum B^{+} $-$-

R 6.4.2 For $\mathbb{M}(r_2) = 3$, if ther onter neiglinor of r_1 is $4 - \infty = mi -$ poor wertex, then r_i gets if frum fa, ifrom 4-fars sasal if from 8^{+}-fioce. If the outer nighloor of ns is nut 4-senti-pocer vertex, then vy gets $\frac{1}{-}$ trom fa und &f frum 4 -fiece sad ? from s^{+}-fare uppose to v be a 4-full-poor vertios in which $f_1 = [$ vir $x i y_1$ |. $f_3 =$ sayv | wad f_2 und f_1 ave 8^{+}-fines will $d(v_2) = M(r_1) = 3$

7.1 Lat = be a 4-full-poor I weter. Then v ges of from toch $8 + -$ fure and it sambla 3 to v_1 wall r_2.

3. **R. 7.1.1** For df$(v_1) = d(v_1) - 3$, both v_1 aud v_1 gt $\frac{1}{2}$ from 4-fice f_1 und f_2 sond $\frac{1}{2}$ to 3 -verter sond $\frac{1}{2}$ to 4^+-verters.

7.2 Lut e be is 4-full-poor II wetex and r_1 in incsbent wirh 3-[act and r_4 bs incidond with 4 -lace. Thent v gets $\frac{7}{}$ from ewch 5^+-fact and it sombe if to v_1 and to to r_4

R. 7.2 . 1 For $d(v_2) = 3$, mo gets $\frac{1}{2}$ from 4-furs and $\frac{2}{}$ from 8^+-fact sad then it gets $\frac{2}{7}$ from v. verlex, them v_a gets t lrum f_a of from 4 -foce and of from 4-fice sad of from s^+-fare adal then f Erom v.

7.3 Lat v be a 4 -full-poor III vertex. Then v gets f from earh 8^+- Lace und it sasale $\frac{2}{3}$ ta both v_1 and ε_4.

R. 7.3.1 For $A^2(n) - d(m_1) = 3$, ir the onder nighlors of r_1 and v_4 is 4-samb-pour vertiovs, thim both of v_1 and v_4 get 1 from the outer sovighbes of v_1 and r_1 wre mod 4-bomi-pocer wrticis. then th und e_4 get 1 from f_1 sad f_3 sad &t trom 4 -fare aral ? from 8^+ $-$Gurs atal then of from v

R 8. Suppoces to v is $T^{2(v)}$ $-$ verter.

We dediuce induction $5 \times d(m) \geq 3$.

R. 8.1. T^3 $-$ tvrikx.

Let $f = |vv_2v_2|$ and v be 3-verterx ifceident with 4-fare hund 8^+-fice. If r is a T^3 vorlex, then v gots of from 8^+-fice and 1 frum t-face. Thern f sotuls 9 to v :

R. 8.2. T^2 - vxrikx.

If v is T^4-vortex inciblent with one 4 -fuce and ars 8^+ $-$fuce, thest eart 3-fices.

R. 8.3. T^{Th} – wriks

Let $f_1 = |vv_1v_2|$ und $f_2 = |v_2r_1|, v$ gets If from inach 5^+-fact and if from 4-fwoe. Then = samale to to ewch 3-fare.

R. 8.A. $T^{\mathrm{stex)}}$ – wertex

R 8.4.1 Lat = be a T^{divel} -vertex soch that n is even aul $n \geq 6$. v gets 2 from ewh 8^+-fince sasd ffrom 4-fure In grastal v

R. 8.4.2 Lat v be a $T^{-(v)}$-vertex such that $d(v)$ is call and $\mathbb{A}(p) \geq 7$. Hete v in incsbont wilh $\left(\frac{de+}{2} - i\right)$ 8^+-fauce whare $d(v) = 4r + 3, r - 1,2, \ldots, n$ und $d(v) \geq 7$ aul ingident with $\left|\frac{4v}{2}\right|$ 3-liute sad two 5^+ $-$fiest Thuse v gets of fromi each 8^+-fare, of from 4-face and 3 from tath 5^+-Sime. Lh guaral fot $d(v) = 4n + 3, n = 1,2, \ldots$, und $d(v) \geq 7, v$ sutuls $\frac{524]+-194}{\tan^2(p)}$ to stach 3-[ave. (R 8.4.3) Lat v be a T^{vel} -vertex soch that $d(v)$ is oald and $d(v) \geq 9$. Hote = is incident wilh $\left(\frac{dv}{4} - \frac{5}{4}\right)$ 8^+ $-$fice where $d(v) = 4n + 5, n = 1,2, \ldots, n$ and $d(v) \geq 9$ asd incilifot with $\left|\frac{4\sqrt{2}}{2}\right|$ 3-fice adal two $5^+ - 5$ thes. Thim v gets + from varh 8^+-fince, f trom ench 4 -fice sud ? Iroum earh is $^+$ $-$Inoe. In gexarral foe $d(v) = 4n + 5, n - 1,2, \ldots$, und $d(v) \geq 9, v$ thes r gets $\frac{1}{4}$ from 4 -lace, from 6^+-face and $\frac{9}{}$ from 8^+-fhoe and amale 1 to 3 -lace:

R 10. Oehurwises, ir v is rast a pour vetex in whidh $f = |v_1 - v_2, v_2| = (3,4,5)$-face, thes / gess 1 lrom 4 -vertex and $\frac{1}{2}$ frum 5-vertex and them it sunds $\frac{9}{}$ to $n2$. 0) Fer all $x \in V \cup F$. Lat $= EV(G)$ sad $f \in F(G)$. The peroal caa be cutupleted

with $d(x)$ for $allx \in V \cup F$. Iot $=\in V(G)$ aul $f \in F(G)$. Since $d(e) \geq 3$. If df(v) $= 4$, tr **R**1 sal **R** 2, then vis a 4-light wetex with $f - (3,4,5^+)$-fare So, $\mathrm{ch}'(v) = \mathrm{ch}(v) + 2 \times \frac{1}{2} + \frac{1}{2} - \frac{3}{2} \times 4 - 7 + 2 \times \frac{1}{2} + \frac{1}{4} - \frac{3}{2} - 0$by**R** 2.1. Coutinuomly, if $d(v)^2 - 3^2$ by R. 2.1 aud R. 5, thon $f = (3.4, 5^+)$-face und the 3-wsters is 3-full-poor vettex. 5Sa, $\hat{N}'(e) = d(v) + \frac{10}{3} + \frac{1}{2} - 0$ by **R** 2.1udidd $\mathrm{N}'(v) = d(v) + \frac{10}{3} + \frac{1}{2} + \frac{3}{2} > 0$RR.

If $f = [v_1 v_{yea}] = (3,4,5)$ ly R. 1 sad R. 3 sad loy Lumat 2.8, then $M(v) = d(v) + 2 \times \frac{1}{2} + \frac{3}{2} - 0$ ly **R** 3.1. And thest fint $d(v) = 4, df'(v) = \mathrm{cl}(v) + \frac{1}{2} + \frac{1}{5} - \frac{1}{2} - -1 + \frac{1}{2} + \frac{1}{2} \geq 0$**R** 3.2. Morevener, fir $d(v) - 5$. $dP(v) = \mathrm{ch}(v) + \frac{2}{5} + 2 \times \frac{5}{2} - \frac{1}{2} + \frac{1}{3} + 2 \times \frac{5}{6} - \frac{2}{2} \geq 0$ R. 3.3 . If $d(v) = 3$ and by R 1 uni R 4.5 so, we lauve dh' $(v) = m(v) + 3 \times 4 - \frac{3}{2} \times 3 - 7 + 3 \times 2 - 0$ by R 4.1. By Cocollary 2.12 if $d(x) = d(y) = d(z) - 3$ atul thery are 3-semipoor vestios, then $d(x;) \geq 5. So, N'(v) = d(v) + 3 \times \frac{1}{2} + 3 \times \frac{1}{2} - -\frac{5}{2} + \frac{4}{2} = 0$ by R 4.2. If d(v) -3 wad $f = |vv_1v_2| = (3,4,4^+)$wad $N(v) = (v_1, v_2, v_3)$ by R 1 and R 5 and ly Ievuma 2.13, then v in a 3-full-poor wettex. Sos, $\mathrm{c}'(v) = d(v) + \frac{1}{2} + \frac{1}{2} -$

$\frac{\pi}{3}--\frac{4}{2}+\frac{5}{2}-0$ by R 5. Thun, if i_2 is a 4 -poor vertex, ben ive is incilintat with 4-fure, 6^+-face and 8^+-bare. So, for $d(v)\geq 4, d'(v)=m(v)+t+4+\frac{\pi}{3}-1\geq 0$ by **R**9mul**R**. . Here, for $3-$ Bare, $\boldsymbol{k}'(f)=d(f)+t+\frac{1}{2}+1>0$**R** 3.2 und **R**5 acal **R9**. with $d(v_2)=d(v_4)-3$, then v is a 4 -bomb-poos vertex hy **R**1 wad **R** 6. If v bi is 4-semi-poor varter I, then $N'(e)=d(v)+\frac{1}{1}+2\times\frac{1}{2}-\frac{1}{2}--1+\frac{1}{1}+\frac{1}{1}-\frac{1}{2}>0$ by **R** 6.1. For $d(v_1)-3$, we mutst have $d(v_2)\geq 4$. So, $N'(r_1)=d(r_1)+\frac{t}{t}+t+\frac{i}{2}-0$ by **R** 6.1.1 and R 9 . Tloen $f=[r_1v_1]$, $N'(f)=M(f)+\frac{3}{2}+1-k>0$ by R. 6.1. R. 6.1.1 und R. 9. Fir $d'(v_1)-3$. if r_1 is incident with $f=(3,4,5)-$ Lars, then $N'(r_4)=\mathrm{ch}(v_i)+3+\frac{2}{2}+\frac{2}{2}>0$ $t+2\times t-\frac{3}{2}--1+\frac{1}{4}+\frac{2}{4}-\frac{2}{2}-0$ by **R** $\cdot$ 6.2. For $d(v_4)=3$, ir the onter usighlour of x_1 in 4-ormi-poot vetex, thes $N'(x_1)-d(x_4)+3+4+\frac{2}{2}\geq 0$ by R. 6.2.2. For $d(x_4)=3$, if the outer wisflahor of x_4 in 4-full-poocr vertioc.

If v is is 4 -smai-poor vertex III, then $\mathrm{ch}'(v)=\mathrm{d}(e)+\frac{1}{2}+2\times 2-\frac{1}{2}=-1+\frac{1}{2}+\frac{2}{4}-\frac{2}{2}>0$ by R 6.3. For $d(v_1)-3$, we mist have $d(v_2)\geq 4$. So, $N'(v_1)-d(m_1)+\frac{z}{3}+\frac{2}{2}+\frac{1}{3}>0$ by **R** 6.3.1 ama R **R**. Then $f=[=v_1v_y]$, $M(f)=d(f)+\frac{1}{2}+1-\frac{3}{2}>0$ ly **R**. 6.3, R. 6.3.1 sad R. 10. For $d(t_1)-3$. if v_4 is incitlost with $f=(3,4,5)$-Facte, thea $cf'(v_4)=\mathrm{ch}\ (v_a)+\frac{2}{2}+2+9>0$ by R. 6.3 .1 und R 10.

If v is a 4 -owmi-poour wertex IV, thaw $dK'(v)=m(c)+\frac{1}{4}+2\times 9-\frac{3}{2}-\ -1+\frac{2}{3}+\frac{9}{2}-\frac{3}{2}-0$ ly **R** 6.4. For $d(x_y)-3$, if the owter nighbor of $\approx_1$ is 4-acmi-poor vertex, then $\mathrm{ch}'(\mathrm{v}_1)-\mathrm{de}(\mathrm{v}_4)+\frac{1}{4}+\frac{1}{4}+\frac{2}{3}\geq 0$ be **R** $M(r_1)=d(v_4)+1+4+3+4\geq 0$ by R E.A.2 mad R T.1

Fur $d(v)=4$, ir $f_1=|v\nabla_1x_2|, f_1=[vr_2\parallel r_2]$ und f_2 and f_4 are 8^+-ficts If = is a 4 -full-powe vortex 1 , thes $dh'(v)-d(v)+t+2\times t-2\times 3-$
$-1+4-t>0$ by **R** 7.1. For $d(v_1)=d(r_1)=3$, if r_1 und r_a are insbont with $f=(3,4,5)$, then $dh'(v)=m(r)+\frac{1}{2}+\frac{2}{t}+\frac{9}{d}+\frac{2}{y}>0$ by **R** 7.1 .1 mal R 11 (wlare r is sepoesituted by r_1 und v_4). If = Bs is 4 -[itl-poor wirtex II, thon $\mathrm{ch}'(v)=d(c)+\frac{1}{5}+2\times\frac{2}{-2}-\frac{1}{5}--1+\frac{1}{5}+\frac{9}{2}-\frac{3}{5}>0$ by R 7.2. For $d(r_4)=3$. ar the owter tavighoot of r_2 is 4-wami-pose verLiox, then $\mathrm{CM}'(v_1)=d(v_2)+\frac{1}{2}+\frac{1}{5}+\frac{9}{4}+\frac{2}{3}=0$ ly R. 7.2 .2 and R 6.1. For $d(v_4)=3$, ir the owher nwighbor of v_4 is 4-[ull-poser vetwx, then

Fur $W(v)-3$, In R 1 und R. 8. if = Bo inciuluak with Z-fioor, 4-fout Here, v is T^3-virtex aul we cau get n_1 is a 4 -womi-pose wot wx sud $v_2\geq 4$ und $=0\mathrm{cM}(v)=h(v)+\frac{1}{4}+\frac{2}{+}+\frac{2}{2}-0$ ly R. 8.1. R. 6 sad R. 9. Thut $dM(f)=M(f)+\frac{2}{2}+1-\frac{2}{3}>0$ by R. 8.1. R. 6 and R. 9.

For $d(r)=4$, ly **R**1 und **R** 8, if r ins incsbont wirle two 3-farss, was 4-fiwer and une 8^+ $-$fack, thea v is a T^2- vetex. Let $f_1=|vv_2\mathrm{r}_2|$ ithal $f_2=[vv_2\mathrm{r}_4\ |\cdot f_2$ be 4-Gare und f_1 is s^+ $-$fice. $50, \mathrm{ch}'(v)=\mathrm{ch}(v)+\frac{1}{1}+\frac{10}{2}-2\times 2\cdot\frac{2}{2}-0$ by R. 8.2. Lut $f_1=f_1=(3,4,5)$. Ir v is a T^4-vertex, then cli $(f)=d(f)+t+\frac{\pi}{2}-t<0$ by R 8.2, R. 10 or $d'(f)=ch(n)+t+t-t<0$ ly R. 8.2. R. 3.1. So, it is impocsilde that T^{-4}-vertex is mljecent to 3-vortex.

Lemma 3.2 Lat $f_1-[rv_1v_2\ |$ and $f_1-|tr_1v_2|, f_2$ be 4 -farx erod f_1 is und two $5^+-[$ fioe, than v is a T^5- vartex. Lat $f_1=[vv_1v_2$ anal $f_1=[vv_1v_2], f_2$ 0lng**R** B.3. R 9 iud R. 3.1 ur $c'(f)-\mathrm{ch}(f)+\frac{7}{7}+\frac{3}{2}-\frac{1}{2}<0$ by R. 8.2 R 3.1 und R 10. So, it is imposbilhle that T^5-vertex is auljuciot to $3-$ poour vortox. Thus $d'(f)=\mathrm{ch}(f)+\pi+\frac{2}{2}-\frac{1}{r}>0$ln R 8.2, R. 10 and wijuorut to T^a- vortex, thas $f=(5,3.5^+)$-bare

Lemma 3.3 In G, let v be a T^3-verlex in which $f_1=[$ rer $_1v_2$ and $f_2=[$ rav $]_2, f_2$ be 4 -fare aud f_5 be 5^+-fares If a T^5-verlex is efjarout to T^a- virtax, born $f_1=f_2=(5,3,5^+)$- form. Motiver, if = is a $T^{d/4}$-vortex, where $d(z)\geq 6$ und $d(v)$ is mex, by

$$\hat{N}'(v) \geq \mathrm{ch}(v) + \frac{3}{8}\left(\left\lceil\frac{d(v)}{4}\right\rceil\right) + \frac{1}{4}\left(\left\lceil\frac{d(v)}{4}\right\rceil\right) - \frac{\hbar h'(v) - 224}{16d(v)}\left\lceil\frac{d(v)}{2}\right\rceil$$
$$-\frac{3}{2}d(v) - 7 + \left\lceil\frac{3(v)}{32}\right\rceil + \left\lceil\frac{2d[v)}{32}\right\rceil - \frac{53\ \mathrm{h}(v) - 224}{[6\mathrm{f}[(v)}\left\lceil\frac{d(v)}{2}\right\rceil$$
$$= \frac{3N(v) - 224}{32} - \frac{\mathrm{dN}(v) - 224}{16d\{(v)}\left\lfloor\frac{\mathrm{d}\{v)}{2}\right\rfloor$$
$$\geq 0$$

by R. 8.4.1.

If v bi a $T^{-(n)}$-wetex $(A(x) \geq 7, d(\varepsilon) = 4n + 3$, whete $n = 1,2,-)$ in **R** 8.4.2 and hy Corcilhary 2.16. tham

$$\mathrm{cl}'(e) \geq \mathrm{ch}(v) + \frac{3}{8}\left(\frac{d(v)}{4} - \frac{3}{4}\right) + \frac{1}{4}\left(\left\lceil\frac{d(v)}{4}\right\rceil\right) + 2 \times \frac{3}{5} - \left(\frac{\pi 2N(v) - 194}{16N(v)}\right)\left\lceil\frac{d(v)}{2}\right\rceil$$
$$-\frac{3}{2}d(v) - 7 + \frac{3d(v)}{32} + \left\lceil\frac{d(v)}{16}\right\rceil + \frac{6}{5} - \frac{9}{32} - \frac{52d(v) - 194}{[fid(v)}\left(\frac{d(v)}{2}\right]$$
$$-\frac{51d[(c)}{32} + \left|\frac{d(v)}{16}\right) - \frac{973}{160} - \frac{52d(v) - 194}{16af(v)}\left\lfloor\frac{d(v)}{2}\right\rceil$$
$$\leq \frac{26id(v) - 9\pi 3}{160} - \frac{52\ \mathrm{d}(v) - 194}{32}$$
$$-\frac{265d(v) - 9\pi 3}{160} - \frac{266M(v) - 970}{169}$$
$$> 0$$

8.4.3 mad hy Corcillary 2.17, tham

$$\leq \frac{26id(v) - 1018}{160} - \frac{532(v) - 202}{32}$$
$$-\frac{265\ \mathrm{d}(v) - 1018}{160} - \frac{260\ \mathrm{N}(v) - 1010}{160}$$
$$> 0$$

If o is a 4-light wortex, then $f = [(rvyt) = (3,3,4)$-fice by R1 and R2.1 wal

R. . If v_1 und v_2 are 3-full-poos wortions, then $\mathrm{cN}'(f) = ch(f) + 1 + f + \frac{7}{20} - 2d(f) - 7 + if \geq 0$. By Lumima 29 , when $d(f) = 4$, f sunuls t to carh 4-light vertex, क्ष $(\rho) = \mathrm{ch}(f) - 4 \times \frac{1}{2} - 0$ ly R 2.1 und R 1. Suppose by **R.** 3.1, **R**3.2 and R 3.3. By R. 10, if v_1, v_2 wad v_3 are hot poos verticts.

Then $\mathrm{ch}^2(\rho) = \mathrm{ch}(\rho) + \frac{1}{2} + 1 - \frac{1}{n} - 2N(n) - 7 + \frac{1}{7} > 0$.

For $d(\rho) = 4$, by Lemma 2.11, $\hat{N}'(\rho) = d(f) - t - \frac{1}{3} - 2d(f) - 7 - t < 0$ by R. 3.2. R. 4.1 satal R. 6.1. So, Lemma 2.11 is true. und $d(G) \geq 3$, the following lomina be olwiote. This coupletes the proof of Thasorim 1.I.

4. CONCLUSION

Planar graph: A graph that can be embedded in the plane without any edges crossing.

Adjacent triangles or 7-cycles: This means that the graph does not contain any adjacent triangles (cycles of length 3) or 7-cycles (cycles of length 7). In other words, there are no three vertices connected pairwise by edges such that they form a triangle, and there are no cycles of length 7.

(3, 1)-choosable: This refers to a graph coloring property. A graph is said to be (a, b)-choosable if whenever each vertex is assigned a list of at least 'a' colors, and each vertex has at most 'b' neighbors with the same list of colors, then there exists a proper coloring of the graph where each vertex is assigned a color from its list such that no adjacent vertices share the same color.

The conclusion you provided states that every planar graph that does not contain adjacent triangles or 7-cycles is (3, 1)-choosable.

This result likely comes from a deeper proof involving techniques from graph theory and combinatorics. The idea is to show that such graphs can be colored with at most 3 colors in such a way that no adjacent vertices have the same color, given that each vertex has at most 1 neighbor with the same set of available colors.

This kind of result can have applications in various areas, including scheduling problems, network optimization, and other fields where graph coloring plays a role.

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