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A NOVEL ILLUSTRATION OF THE GENERALIZED KRÄTZEL FUNCTION

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ABSTRACT

The ability to combine distributions (generalized functions) with integral transformations has grown to be a very effective tool for solving significant open issues. Investigating a distributional representation of the generalized Krätzel function is the aim of the current work.

Thus, over a certain collection of test functions, a new definition of these functions is developed. Using the classical Fourier transform, this is confirmed. The findings introduce distributions in terms of the delta function, which leads to a novel extension of Krätzel functions. The result of this research is a new version of the generalized Krätzel integral transform. In order to investigate novel identities, the connection between the Krätzel function and the H-function is also investigated.

Keywords: delta function, generalized functions (distributions), slowly increasing test functions, generalized Krätzel function, H-function, and Fourier transformation.

1. INTRODUCTION

Recent research, such as [1,10], address mathematical features of the generalized Krätzel function and integral operators. A revised form of the generalized Krätzel-integral operators is explained with reference to distribution theory components. In these functions are also examined through the Boehmians' Fréchet space. To the best of the author's knowledge, no research has been done on the Krätzel function as a distribution in terms of the delta function in the literature. Inspired by the conversation above, the current work focuses on examining a novel representation of the generalized Krätzel function. It is possible to expand the domain of the generalized Krätzel function [7] from complex numbers to the space of complex test functions by doing this. Clearly, the H-function will yield findings along similar lines when taking into account the connections [5] and [6].

This paper will be organized as follows they provides important test function space preliminary information. The following is how the remaining sections are arranged: The generalized Krätzel function is represented by a new series. There is a twin space called the space of distributions (or generalized functions) that corresponds to each space of test functions. Because these functions have the significant characteristic of embodying solitary functions, consideration of them is essential. As with classical functions, several calculus procedures can be used on these kinds of functions. This subsection uses standard notations that are taken from [Zamanian, A.H. Distribution Theory and Transform. Analysis; Dover Publications: New York, NY, USA, 1987., And Richards, I.; Youn, H. Theory of Distributions: A Non-Technical Introduction; Cambridge University Press:Cambridge, MA, USA; London, UK; New York, NY, USA, 2007.]. Nonetheless, this document uses for the test functions. The delta function is a frequently utilized singular function that must be described for the purposes of this inquiry.

The Krätzel function is defined for $x > 0$ by the integral

$$Z_\rho^\nu(x) = \int_0^\infty t^{\nu-1} e^{-t^\varphi - x/t} dt,$$

where $\rho \in \mathbb{R}$ and $v \in \mathbb{C}$, such that $\Re(v) < 0$ for $\rho \leq 0$ (cf. [1]). For $\rho \geq 1$ the function was introduced by Krätzel as a kernel of the integral transform as follows:

$$\left(\mathcal{K}_v^\rho f\right)(x) = \int_0^\infty Z_\rho^\nu(xt) f(t) dt \ (x > 0).$$

The Krätzel function $Z_\rho^\nu(x)$ is related to the modified Bessel function of the second kind K_ν by the relationship

$$Z_1^\nu(x) = 2x^{\nu/2} K_\nu(2\sqrt{x}).$$

The generalized Krätzel function $D_{\rho,r}^{\gamma,\alpha}(x)$ is given in [2,3] by the following relation:

$$D_{\rho,r}^{\nu,\alpha}(x) = \int_0^\infty t^{\nu-1} [1 + a(\alpha - 1) t^\rho]^{1/(\alpha-1)} e^{-xt^-} dt,$$

where $\rho \in \mathbb{R}, r \in \mathbb{R}^+, \nu \in \mathbb{C}$, and $\alpha > 1$. Kilbas and Kumar considered the special case for $r = 1$ in [2], calculated fractional derivatives and fractional integrals of $D_{\rho,1}^{\gamma,\alpha}(x)$, and obtained a representation using Wright hypergeometric functions. On the other hand the general case is given in [2].

We consider the generalized Krätzel function $Y_{\rho,r}^{\nu}(x)$ defined by the integral

$$Y_{\rho,r}^{\nu}(x) = \int_0^{\infty} t^{\nu-1} e^{-t^p - xt^-} dt,$$

for $x > 0, \rho \in \mathbb{R}, r \in \mathbb{R}^+$, and $\nu \in \mathbb{C}$. The function $Y_{\rho,r}^{\nu}(x)$ is a generalization of the Krätzel function $Z_{\rho}^{\nu}(x)$ since

$$\lim_{r\to 1} Y_{\rho,r}^{\nu}(x) = Z_{\rho}^{\nu}(x).$$

If $a = 1$, then

$$\lim_{\alpha\to 1} D_{\rho,r}^{\nu,\alpha}(x) = Y_{\rho,r}^{\gamma}(x).$$

We give some definitions and inequalities that will be needed. The Turán type inequalities

$$f_n(x) \cdot f_{n+2}(x) - [f_{n-1}(x)]^2 \geq 0,\ n = 0,1,2,\ldots$$

are important and well known in many fields of mathematics [4]. A function $f(x)$ is completely monotonic on $(0,\infty)$, if f has derivatives of all orders and satisfies the inequality

$$(-1)^m f^{(m)}(x) \geq 0$$

for all $x > 0$ and $m \in \mathbb{N}$ [5]. A function $f(x)$ is said to be log-convex on $(0,\infty)$, if

$$f[\alpha x_1 + (1-\alpha)x_2] \leq [f(x_1)]^{\alpha}[f(x_2)]^{1-\alpha}$$

for all $x_1, x_2 > 0$ and $\alpha \in [0,1]$ [5].

Let $p, q \in \mathbb{R}$ such that $p > 1$ and $1/p + 1/q = 1$. If f and g are real valued functions defined on a closed interval and $|f|^p, |g|^q$ are integrable in this interval, then we have

$$\int_a^b |f(t)g(t)|dt \leq \left[\int_a^b |f(t)|^p dt\right]^{1/p} \left[\int_a^b |g(t)|^q dt\right]^{1/q}.$$

The following inequality is due to Mitrinovic et al.[6] Let f and g be two functions which are integrable and monotonic in the same sense on $[a,b]$ and p is a positive and integrable function on the same interval, then the following inequality holds true:

$$\int_a^b p(t)f(t)dt \int_a^b p(t)g(t)dt \leq \int_a^b p(t)dt \int_a^b p(t)f(t)g(t)dt,$$

if and only if one of the functions f and g reduces to a constant. The Mellin transform of the function f is defined by

$$\mathcal{M}\{f(x);s\} = \int_0^{\infty} x^{s-1} f(x)dx$$

when $\mathcal{M}\{f(x);s\}$ exists. The Mellin transform of the generalized Krätzel function is given by Kilbas and Kumar in [2].

The Laplace transform of the function f is defined by

$$\mathcal{L}\{f(x);s\} = \int_0^{\infty} e^{-sx} f(x)dx$$

provided that the integral on the right-hand side exists. The Liouville fractional integral is defined by

$$(\mathcal{F}_-^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} (t-x)^{\alpha-1} f(t)dt$$

and its derivatives $\mathcal{F}_-^{\alpha}$ and $\mathcal{D}_-^{\alpha}$ are

$$(\mathcal{D}_-^{\alpha} f)(x) = \left(-\frac{d}{dx}\right)^{[\Re(\alpha)]+1} \left(\mathcal{I}_-^{1-\alpha+[\Re(\alpha)]} f\right)(x) = \frac{1}{\Gamma(1-\alpha+[\Re(\alpha)])} \int_x^{\infty} (t-x)^{-\alpha+[\Re(\alpha)]} f(t)dt,$$

where $x > 0, \alpha \in \mathbb{C}$, and $\Re(\alpha) > 0$ [7]. We introduce new operators

$$\begin{aligned} \mathcal{D}_\lambda^v &:= -rx\mathcal{D}_-^{\lambda+1} + (\lambda r - v)\mathcal{D}_-^\lambda, \\ \mathcal{J}_\lambda^v &:= r^2x^2\mathcal{D}_-^{2\lambda+2} + rx(2v - 3\lambda r - r)\mathcal{D}_-^{2\lambda+1} \\ &\quad +(v - r\lambda)(v - 2r\lambda)\mathcal{D}_-^{2\lambda}, \end{aligned}$$

where $v \in \mathbb{C}$ and $\lambda > 0$.

1.1. Distributions and Test Functions

The space of distributions, also referred to as generalized functions, is the dual space that corresponds to each space of test functions. Because these functions have the significant characteristic of embodying solitary functions, consideration of them is essential. As with classical functions, several calculus procedures can be used on these kinds of functions. The standard notations utilized in this subsection. However, the notation φ is used for test functions throughout this manuscript. For the requirements of this investigation, the delta function, which is a commonly used singular function, needs to be mentioned. For any test function $\varphi(\omega) \in \mathcal{D}$, the delta function is defined by

$$\langle \delta(t - \omega), \varphi(t) \rangle = \varphi(\omega)(\forall \wp \in \mathcal{D}, \omega \in \mathbb{R}),$$

$$\text{and}\delta(-\mathrm{t}) = \delta(\mathrm{t}); \delta(\omega \mathrm{t}) = \frac{\delta(\mathrm{t})}{|\omega|}, \text{ where } \omega \neq 0$$

An ample discussion and explanation of distributions (or generalized functions) was presented in five different volumes by Gelfand and Shilov. Functions with compact support and that are infinitely differentiable, as well as quickly decaying, are commonly used as test functions. The spaces containing such functions are denoted by $\mathcal{D}$ and $\mathcal{S}$, respectively. Obviously, the corresponding duals are the spaces $\mathcal{D}'$ and $\mathcal{S}'$. A noteworthy fact about such spaces is that $\mathcal{D}$ and $\mathcal{D}'$ do not hold the closeness property with respect to the Fourier transform, but $\mathcal{S}$ and $\mathcal{S}'$ do. In this way, it is remarkable that the elements of $\mathcal{D}'$ have Fourier transforms that form distributions for the entire function space Z whose Fourier transforms belong to $\mathcal{D}$. Further to this explanation, it is notable that as the entire function is nonzero for a particular range $\omega_1 < s < \omega_2$, but zero otherwise, the following inclusion of the abovementioned spaces holds:

$$\mathcal{Z} \cap \mathcal{D} \equiv 0; Z \subset \mathcal{S} \subset \mathcal{S}' \subset Z'; \mathcal{D} \subset \mathcal{S} \subset \mathcal{S}' \subset \mathcal{D}'$$

More specifically, space Z comprises the entire and analytic functions sustaining the subsequent criteria

$$|s^q \varphi(s)| \leq C_q e^{\eta|\theta|}; (q \in \mathbb{N} \setminus \{0\}).$$

Here and in the following, the numbers η and C_q are dependent on φ, and $\mathbb{N}$ denotes the set of natural numbers. where $\mathcal{F}$ denotes the Fourier transform.

$$\mathcal{F}[\mathrm{e}^{\alpha \mathrm{t}}; \theta] = 2\pi\delta(\theta - \mathrm{i}\alpha);$$

$$g(\mathrm{s} + \mathrm{b}) = \sum_{\mathrm{j}=0}^{\infty} g^{(\mathrm{j})}(\mathrm{s}) \frac{\mathrm{b}^\mathrm{j}}{\mathrm{j}!}, \forall g \in \mathcal{Z}';$$

$$\delta(\mathrm{s} + \mathbf{b}) = \sum_{\mathrm{j}=0}^{\infty} \delta(\mathrm{j})(\mathrm{s}) \frac{\mathrm{b}^\mathrm{j}}{\mathrm{j}!}, \text{ where } \langle \delta^{(\mathrm{j})}(\mathrm{s}), \varphi(\mathrm{s}) \rangle = (-1)^\mathrm{j} {}_\varphi{}^{(j)}(0);$$

$$\delta(\omega_1 - \mathrm{s})\delta(\mathrm{s} - \omega_2) = \delta(\omega_1 - \omega_2).$$

Some examples include $\sin(t), \cos(t)$, sinht, and cosh t, whose Fourier transforms are delta (singular) functions. Relevant detailed discussions about such spaces can be found in Zamanian 1987 & Richards 2007.

2. RESULTS

2.1. New Representation of Generalized Kratzel Function

In this section, the results are computed as a series of complex delta functions, and discussion about its rigorous use as a generalized function over a space of test functions is provided in the next section.

Theorem 1. The generalized Kratzel function has the following representation in terms of complex delta functions.

$$\mathrm{Z}_{\sigma,\rho}^{\mathrm{a,b}}(s) = 2\pi \sum_{\mathrm{n,r}=0}^{\infty} \frac{(-a)^n(-b)^r}{\mathrm{n}!\,\mathrm{r}!} \delta(\theta - \mathrm{i}(v + \sigma\mathrm{n} - \rho\mathrm{r})).$$

Proof. A replacement of $\mathrm{t} = \mathrm{e}^\omega$ and $\mathrm{s} = \mathrm{v} + i\theta$ in the integral representation of the generalized Krätzel function as given in (6) yields the following:

$$\mathrm{Z}_{\sigma,\rho}^{\mathrm{a,b}}(\mathrm{s}) = \int_{-\infty}^{\infty} \mathrm{e}^{\omega(v+i\theta)} \exp(-\mathrm{a}\mathrm{e}^{\sigma\omega}) \exp(-\mathrm{b}\mathrm{e}^{-\rho\omega}) \mathrm{d}\omega.$$

Then, the involved exponential function can be represented as

$$\exp(-\mathrm{a}e^{\sigma\omega})\exp(-\mathrm{b}e^{-\rho\omega}) = \sum_{\mathrm{n}=0}^{\infty}\frac{(-\mathrm{a}e^{\sigma\omega})^{\mathrm{n}}}{\mathrm{n}!}\sum_{\mathrm{r}=0}^{\infty}\frac{(-\mathrm{b}e^{-\rho\omega})^{\mathrm{r}}}{\mathrm{r}!}.$$

Next, leads to the following:

$$Z_{\sigma,\rho}^{a;b}(s) = \int_{-\infty}^{\infty} e^{ix\theta}\sum_{n,r=0}^{\infty}\frac{(-a)^n(-b)^r}{n!\,r!}e^{(v+\sigma\sigma-\rho r)\omega}d\omega,$$

which gives

$$Z_{\sigma,\rho}^{a,b}(s) = \sum_{n,r=0}^{\infty}\frac{(-a)^n(-b)^r}{n!\,r!}\int_{-\infty}^{\infty} e^{i\omega\theta\theta}e^{(v+\sigma\sigma-\rho r)\omega}d\omega.$$

The actions of summation and integration are exchangeable because the involved integral is uniformly convergent. An application of identity and produces the following:

$$\int_{-\infty}^{\infty} e^{i\theta x}e^{(v+\sigma n-\rho r)\omega}dx = \mathcal{F}\left[e^{(v+\sigma n-\rho r)\omega};\theta\right] = 2\pi\delta(\theta - i(v+\sigma n-\rho r)).$$

Corollary 1. The generalized Kratzel function has the following series form.

$$Z_{\sigma,\rho}^{a,b}(s) = 2\pi\sum_{n,r,p=0}^{\infty}\frac{(-a)^n(-b)^r(-i(v+\sigma n-\rho r))^p}{n!\,r!\,p!}\delta^{(p)}(\theta).$$

Proof. Equation can be obtained by considering the following combination of Equations

$$\delta(\theta - i(v+\sigma n-\rho r)) = \sum_{p=0}^{\infty}\frac{(-i(v+\sigma n-\rho r))^p}{p!}\delta^{(p)}(\theta).$$

Next, making use of this relation leads to the required form.

Corollary 2. The generalized Kratzel function has the following series form.

$$Z_{\sigma,\rho}^{a;b}(s) = 2\pi\sum_{n,r=0}^{\infty}\frac{(-a)^n(-b)^r}{n!\,r!}\delta(s+\sigma n-\rho r).$$

Proof. Equation (23) can be rewritten as follows:

$$\begin{aligned}\int_{-\infty}^{\infty} e^{i\theta x}e^{(v+\sigma n-\rho r)\omega}d\omega &= \mathcal{F}\left[e^{(v+\sigma n-\rho r)\omega};\theta\right] = 2\pi\delta(\theta - i(v+\sigma on-\rho r))\\ &= 2\pi\delta\left[\frac{1}{i}(i\theta + (v+\sigma on-\rho r))\right]\\ &= 2\pi|i|\delta(v+i\theta+\sigma r-\rho r) = 2\pi\delta(s+\sigma n-\rho r).\end{aligned}$$

Next, making use of this relation in leads to the required form.

Theorem 2. The generalized Kratzel function holds the subsequent properties as a distribution

(i) $\langle Z_{\sigma,\rho}^{a\,b}(s), \wp_1(s)+\wp_2(s)\rangle = \langle Z_{\sigma,\rho}^{a;b}(s), \wp_1(s)\rangle + \langle Z_{\sigma,\rho}^{a;b}(s), \wp_2(s)\rangle; \forall\wp(s)\in\mathcal{Z}$

(ii) $\langle c_1 Z_{\sigma,\rho}^{a,\,b}(s), \varphi(s)\rangle = \langle Z_{\sigma,\rho}^{a;b}(s), c_1\varphi(s)\rangle; \forall\varphi(s)\in\mathcal{Z}$

(iii) $\langle Z_{\sigma,\rho}^{a,\,b}(s-\gamma), \varphi(s)\rangle = \langle Z_{\sigma,\rho}^{a;b}(s), \varphi(s+\gamma)\rangle; \forall\varphi(s)\in\mathbb{Z}$

(iv) $\langle Z_{\sigma,\rho}^{a;b}(c_1 s), \wp(s)\rangle = \left\langle Z_{\sigma,\rho}^{a;b}(s), \frac{1}{c_1}\varphi\left(\frac{s}{c_1}\right)\right\rangle; \forall\varphi(s)\in\mathcal{Z}$

(v) $\langle Z_{\sigma,\rho}^{a;b}(c_1 s-\gamma), \varphi(s)\rangle = \left\langle Z_{\sigma,\rho}^{a;b}(s), \frac{1}{c_1}\varphi\left(\frac{s}{c_1}+\gamma\right)\right\rangle; \forall\wp(s)\in\mathcal{Z}$

(vi) $\psi(s)Z_{\sigma,\rho}^{a;b}(s)\in Z$ is a distribution over Z for any regular distribution $\psi(z)$.

(vii) For $b=0, Z_{\sigma,\rho}^{a;0}(s) = s\sigma_{\sigma,\rho}^{a;0}(s)$ iff $\varphi(s-1) = s\varphi(s)$, where $\varphi\in\mathcal{Z}$

(viii) $\langle Z_{\sigma,\rho}^{a\,b}(m)(s), \varphi(s)\rangle = \sum_{n,r=0}^{\infty}\frac{(-a)^n(-b)^r}{n!r!}(-1)^m\rho^m(-\sigma n+\rho r); m=0,1,2,\ldots; \forall\varphi(s)\in\mathcal{Z}$

(ix) $Z_{\sigma,\rho}^{a;b}(\omega_1-s)Z_{\sigma,\rho}^{a;b}(s-\omega_2) = (2\pi\exp(-a-b))^2\delta(\omega_1-\omega_2)\big); \forall\wp(s)\in\mathcal{Z}$

(x) $\langle\mathcal{F}[Z_{\sigma,\rho}^{a;b}(s)], \wp(s)\rangle = \langle Z_{\sigma,\rho}^{a;b}(s), \mathcal{F}[\wp](s)\rangle; \forall\wp(s)\in\mathbb{Z}$

(xi) $\langle\mathcal{F}[Z_{\sigma,\rho}^{a;b}(s)], \mathcal{F}[\wp(s)]\rangle = 2\pi\langle Z_{\sigma,\rho}^{a;b}(v), \wp(-v)\rangle, v=\Re(s); \forall\wp(s)\in\mathcal{Z}$

(xii) $\left\langle \overline{\mathcal{F}[\mathbf{Z}_{\sigma,\rho}^{a;b}(s), \mathcal{F}[\wp(s)]} \right\rangle = 2\pi\langle \mathbf{Z}_{\sigma,\rho}^{a;b}(v), \wp^T(v)\rangle$, where $\overline{\wp(-s)} = \wp^T(s); \forall\wp(s) \in \mathcal{Z}$

(xiii) $\langle \mathcal{F}[Z_{\sigma,\rho}^{a;b}(s)], \bar{\mathcal{F}}[\wp(s)]\rangle = 2\pi\langle Z_{\sigma,\rho}^{a;b}(v), \wp^T(v)\rangle; \forall\varphi(s) \in \mathcal{Z}$

(xiv) $\langle \overline{\mathcal{F}[Z_{\sigma,\rho}^{a;b}(s)]}, \overline{\mathcal{F}[\rho(s)]}\rangle = 2\pi\langle [Z_{\sigma,\rho}^{a;b}(v)], [\wp(v)]\rangle; \forall\wp(s) \in \mathcal{Z}$

(xv) $\mathcal{F}[Z_{\sigma,\rho}^{a;b}(m)(s)] = [(-it)^m Z_{\sigma,\rho}^{a;b}(s)]; m = 0,1,2, \ldots; \forall_{\wp}(s) \in \mathcal{Z}$

(xvi) $Z_{\sigma,\rho}^{a;b}(s + c_1) = \sum_{n=0}^{\infty} \frac{(c_1)^n}{n!} Z_{\sigma,\rho}^{a;b}(n)(s); \forall\wp(s) \in \mathcal{Z}$

where c_1, γ, and c_2 are arbitrary real or complex constants.

3. DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER

In this section, we show that $Y_{\rho,r}^{v}(x)$ is the solution of differential equations of fractional order.

Theorem 3. If $\alpha, v \in \mathbb{C}, \Re(\alpha) > 0$, and $\rho > 0$, then the following identity holds true:

$$\left(\mathcal{J}_{-}^{\alpha} Y_{\rho,r}^{v}\right)(x) = Y_{\rho,r}^{v+r\alpha}(x).$$

Proof.

$$\begin{aligned}
&\left(\mathcal{G}_{-}^{\alpha} Y_{\rho,r}^{v}\right)(x) \\
&= \frac{1}{\Gamma(\alpha)} \int_x^{\infty} (t-x)^{\alpha-1} dt \int_0^{\infty} u^{v-1} e^{-u^{\rho} - tu^{\gamma}} du \\
&= \int_0^{\infty} u^{v-1} e^{-u^{\rho}} \left(\frac{1}{\Gamma(\alpha)} \int_x^{\infty} (t-x)^{\alpha-1} e^{-tu^{\mathrm{T}}} dt \right) du \\
&= \int_0^{\infty} u^{v-1} e^{-u^{p}} \left(\mathcal{F}_{-}^{\alpha} e^{-tu^{r}}\right)(x) du \\
&= \int_0^{\infty} u^{\gamma-1} e^{-u^{\rho}} e^{-xu^{-r}} u^{r\alpha} du \\
&= \int_0^{\infty} u^{\gamma+r\alpha-1} e^{-u^{\rho}} e^{-xu^{-r}} du = Y_{\rho,r}^{v+r\alpha}(x).
\end{aligned}$$

Theorem 4. If $\alpha, v \in \mathbb{C}, \Re(\alpha) > 0$, and $\rho > 0$ then we have

$$\left(\mathcal{D}_{-}^{\alpha} Y_{\rho,r}^{\gamma}\right)(x) = Y_{\rho,r}^{v-r\alpha}(x).$$

Proof.

$$\begin{aligned}
&\left(\mathcal{D}_{-}^{\alpha} Y_{\rho,r}^{v}\right)(x) = \left(-\frac{d}{dx}\right)^{[\Re(\alpha)]+1} \left(\mathcal{F}_{-}^{1-\alpha+[\Re(\alpha)]} Y_{\rho,r}^{\gamma}\right)(x) \\
&= \left(-\frac{d}{dx}\right)^{[\Re(\alpha)]+1} Y_{\rho,r}^{v+r(1-\alpha+[\Re(\alpha)])}(x) \\
&= \left(-\frac{d}{dx}\right)^{[\Re(\alpha)]+1} \int_0^{\infty} t^{v+r(1-\alpha+[\Re(\alpha)])-1} e^{-t^{\rho} - xt^{r}} dt \\
&= \int_0^{\infty} t^{v+r(1-\alpha+[\Re(\alpha)])-1} e^{-t^{\rho}} \left(-\frac{d}{dx}\right)^{[\Re(\alpha)]+1} \\
&\cdot \left(e^{-xt^{t}}\right) dt = \int_0^{\infty} t^{v+r(1-\alpha+[\Re(\alpha)])-1} e^{-t^{\rho}} \\
&\cdot \frac{1}{t^{r([\Re(\alpha)]+1)}} e^{-xt^{-r}} dt = \int_0^{\infty} t^{v-\alpha r-1} e^{-r^{\rho}} e^{-xt^{-r}} dt \\
&= Y_{\rho,r}^{v-r\alpha}(x).
\end{aligned}$$

Corollary . If α, β, and $v \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0$, and $\rho > 0$, then we have

$$\begin{aligned}
\left(\mathcal{D}_{-}^{\alpha} \mathcal{F}_{-}^{\beta} Y_{\rho,r}^{v}\right)(x) &= \left(\mathcal{F}_{-}^{\beta} \mathcal{D}_{-}^{\alpha} Y_{\rho,r}^{v}\right)(x) \\
&= Y_{\rho,r}^{\gamma+\beta-\alpha+(1-r)(1+[\mathrm{Re}\,(\alpha)])}.
\end{aligned}$$

Theorem 5. If $v \in \mathbb{C}$ and $\rho > 0$, then the following identity holds true:

$$\mathcal{L}_{\rho}^{v} Y_{\rho,r}^{v}(x) = -\rho Y_{\rho,r}^{v+(1-r)\rho}(x).$$

Proof. Applying (17) to (5), we get

$$\begin{aligned}\mathcal{L}_\rho^\nu Y_{\rho,r}^\nu(x) &= -rx\mathcal{D}_-^{\rho+1}Y_{\rho,r}^\nu(x) + (\rho r - v)\mathcal{D}_-^\rho Y_{\rho,r}^\nu(x)\\ &= -rxY_{\rho,r}^{\nu-(\rho+1)r}(x) + (\rho r - v)Y_{\rho,r}^{\nu-\rho r}(x)\\ &= -rx\int_0^\infty t^{\nu-(\rho+1)r-1}e^{-r^{\rho-}-xt^T}dt\\ &+(\rho r - \nu)\int_0^\infty t^{\gamma-\rho r-1}e^{-r^\rho-xt^{-t}}dt\\ &= -\int_0^\infty t^{\nu-\rho r-1}\left(r\frac{x}{t^r} + v - \rho r\right)e^{-r^\rho-xt^-}dt.\end{aligned}$$

Using the formula

$$(t^{\nu-\rho r}e^{-xt^-})' = t^{\nu-\rho r-1}(\nu - \rho r + xrt^{-r})e^{-xt^r}$$

and applying the integration by parts, we find

$$\begin{aligned}\mathcal{L}_\rho^\nu Y_{\rho,r}^\nu(x) = &\quad -\int_0^\infty (t^{\nu-\rho r}e^{-xt^-})'e^{-r^\rho}dt\\ = &\quad -t^{\nu-\rho r}e^{-xt^-}e^{-r^\rho}\Big|_0^\infty\\ &\quad +\int_0^\infty t^{\nu-\rho r}e^{-xt^-}\left(-\rho t^{\rho-1}e^{-t^\rho}\right)dt\\ = &\quad 0-\rho\int_0^\infty t^{\nu-\rho r+\rho-1}e^{-xt^-}e^{-r^p}dt\\ = &\quad -\rho Y_{\rho,r}^{\gamma+(1-r)\rho}(x).\end{aligned}$$

4. CONCLUSION

One potent approach to solving difficult and enduring problems in mathematics is the integration of distributions (generalized functions) with integral transformations. This work has focused on investigating a distributional form of the generalized Krätzel function in order to increase its applicability and improve our comprehension of its characteristics.A new definition of these functions across a certain set of test functions has been developed through thorough examination. Through the lens of the classical Fourier transform, the validity of this definition has been verified, confirming the coherence and usefulness of the suggested method.Furthermore, by bringing distributions into the context of the delta function, this study has produced important new insights and opened the door to a brand-new Krätzel function extension. Consequently, a new version of the generalized Krätzel integral transform has been developed, which provides improved mathematical analysis and problem-solving skills.Moreover, via delving into the complex relationship between the Krätzel function and the H-function, this research has revealed new identities and relationships that have improved our understanding of both mathematical objects and their interactions. Overall, this work opens up new directions for investigation and advances our understanding of integral transformations and distribution theory. It is a major contribution to the field of mathematics. The approaches and results discussed here have the potential to handle a broad range of mathematical difficulties and stimulate additional research into the complex field of mathematical analysis.

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