INTERNATIONAL JOURNAL OF PROGRESSIVE RESEARCH IN ENGINEERING MANAGEMENT AND SCIENCE (IJPREMS)
(Int Peer Reviewed Journal)
Vol. 05, Issue 04, April 2025, pp : 2590-2595
www.ijprems.com
editor@ijprems.com
e-ISSN : 2583-1062
Impact Factor : 7.001

SYSTEM OF FRACTIONAL NONLINEAR BOUNDARY VALUE PROBLEM INVOLVING RIEMANN-LIOUVILLE DERIVATIVE

J. A. Nanaware[1], K. A. Shanediwan[2], M. S. Panchal[3]

[1,2]Department of PG Studies and Research in Mathematics, Shrikrishna Mahavidyalaya, Gunjoti, Dist. Dharashiv-413606(M.S.), India.

[3]SNBP School and Junior College, Kokane Chowk, Rahatani, Pune-27 (M.S.), India.

jag_skmg91@rediffmail.com, [2]shanediwankalim@gmail.com,

DOI: https://www.doi.org/10.58257/IJPREMS40511

ABSTRACT

In this paper, Green's function is defined to study an existence and uniqueness of solution for system of fractional nonlinear boundary value problems. Banach contraction principle is also applied to study the problem.

Keywords: Riemann-Liouville fractional derivative, Green's function, Existence and uniqueness of solution, Banach contraction principle.

1. INTRODUCTION

The theory of fractional differential equations attracted many researchers during last two decades due to wide range of applications in applied sciences, economics, engineering and technology etc. For more details reader may refer [7, 9, 12]. The existence, uniqueness and stability results were studied in [6, 13] using Green function, fixed point theorems, monotone iterative techniques and Lie group symmetry etc. Boundary value problems for various types of fractional value problems have been studied by many authors [**1**]-[5] and [10]-[16]. The existence and uniqueness of solutions for a class of fractional nonlinear boundary value problems under mild assumptions was studied by Bachar et. al.[2] using Banach contraction principle.

In 2017, Zou et al.[16] studied uniqueness result for the solution of the following nonlinear boundary value problem

$D^\alpha v(x) + f(x,v(x)) = 0, \qquad 2 < \alpha \le 3, \qquad x \in (0,1),$

$v(0) = v^0(0) = v(1) = 0.$

Recently, Bachar et al. [2] obtained existence and uniqueness results for a class of fractional nonlinear boundary value problems under mild assumptions.

Many physical phenomena are modeled as system of fractional differential equations with various types of conditions that depends upon physical situations. The existence and uniqueness of solutions of system of fractional differential equations with different conditions were studied by researchers [8]. Very few studies on the existence and uniqueness results for system of nonlinear fractional differential equations is rare in the literature. Therefore, investigating this interesting research topic makes our results novel and worthy.

Motivated by aforementioned work, we generalize the results obtained in [2] for the solution of system of following fractional nonlinear boundary value problem:

$$D^\alpha v_i(s) + f_i(s,v_1(s),v_2(s)) = 0, \qquad (1.1)$$

$$v_i(0) = v_i^0(0) = v_i(1) = 0, \qquad i = 1,2,$$

where D^α is the Riemann-Liouville fractional derivative of order α and $f_i \in (0,1)\times \mathbb{R}^2)$ satisfies the following assumptions:

(A_1) $\int_0^1 (1-s)^{\alpha-2}|f_i(s,0,0)|ds < \infty$

(A_2) There exists $q \in C((0,1),[0,\infty))$ such that

$|f_i(s,v_1,v_2) - f_i(s,w_1,w_2)| \le q_i(s)|v_i - w_i|, \qquad \forall\ s \in (0,1), \quad v_i(s),w_i(s) \in R,$

and $0 < M_{q,\alpha} < \infty$, where $0 < \int_0^1 q_i(s)ds < \infty$ and

$$0 < M_{q,\alpha} = \frac{1}{\Gamma(\alpha-1)}\int_0^1 s^{\alpha-1}(1-s)^{\alpha-1}q_i(s)ds.$$

Define the following:

- $h_i(s) = s^{\alpha-1}(1-s), \qquad s \in [0,1], \qquad \alpha \in [2,3].$
- $G_{\alpha i}(s,t)$ be the Green's function of the operator $v_i \to -D^\alpha v_i$ with boundary conditions

- $v_i(0) = v_i^0(0) = v_i(1)$.
- $E = \{a > 0 : \int_0^1 G_{\alpha_i}(s,t)h(t)dt \le ah(s), s \in [0,1]\}$ be nonempty
- And $M_i = \inf E$ (1.2)
- For $a \in R$, $a^+ = \max(a,0)$.
- $C([0,1]) = \{v \in C([0,1])$: there is $\sigma > 0$ such that $|v(s)| \le \sigma h_i(s), s \in [0,1]\}$.

The paper is organized as follows:

In Section 2, we recall some basic definitions from fractional calculus and give some useful preliminary results. In section 3, we prove the existence and uniqueness results for solution of system of fractional nonlinear boundary value problem using Green function.

PRELIMINARIES

In this section, we recall some definitions and results which are used to obtain main results.

Definition 2.1 [9] Euler-Gamma function denoted by $\Gamma(\alpha)$ is defined as

$$\Gamma(\alpha) = \int_0^\infty s^{\alpha-1}e^{-s}ds,$$ where $s^{\alpha-1} = e^{(\alpha-1)\log(s)}$ and $\alpha \in C$.

Definition 2.2 [9] Riemann-Liouville fractional integral of order $\alpha > 0$ of function f is defined as

$$I^\alpha f(s) = \frac{1}{\Gamma(\alpha)} \int_0^s (s-t)^{\alpha-1} f(t)dt.$$

Definition 2.3 [9] Riemann-Liouville fractional derivative of order $\alpha > 0$ of f is defined as

$$D^\alpha f(s) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{ds}\right)^n \int_0^s (s-t)^{n-\alpha-1} f(t)dt,$$

where $n = [\alpha] + 1$, and $[\alpha]$ is the integer part of α.

Lemma 2.1 [14] If $y(s) \in C(0,1) \cap L(0,1)$, then the unique solution of the problem

$^{(}D^\alpha v_i(s) + y(s) = 0, \quad 0 < s < 1, u_i(0) = u^0_i(0) = u_i(1) = 0$

is given by

$$u(s) = \int_0^1 G_{\alpha_i}(s,t)y(t)dt,$$

where $G_{\alpha i}(s,t)$ is the Green function

$$G_{\alpha_i}(s,t) = \frac{1}{\Gamma(\alpha)} \begin{cases} s^{\alpha-1} - (s-t)^{\alpha-1} & for \quad 0 \le t \le s \le 1 \\ s^{\alpha-1}(1-t)^{\alpha-1} & for \quad 0 \le s \le t \le 1. \end{cases}$$

Lemma 2.2 [2] The Green function $G_{\alpha i}(s,t)$ has the following properties:

- $G_{\alpha i}(s,t)$ is nonnegative continuous function on $[0,1] \times [0,1]$
- For all $s,t \in [0,1]$, we have

$H_{\alpha i}(s,t) \le G_{\alpha i}(s,t) \le (\alpha - 1)H_{\alpha i}(s,t)$, (2.1)

where $H_{\alpha_i}(s,t) = \frac{1}{\Gamma(\alpha)} x^{\alpha-2}(1-t)^{\alpha-2} min(s,t)(1-max(s,t))$.

MAIN RESULTS

In this section, we obtain the existence and uniqueness results for the solution of system of fractional nonlinear boundary value problem using Green function. Also, we obtain sequence of iterations that converges to a solution of system of fractional nonlinear boundary value problem.

Lemma 3.1 Let $q_i \in C((0,1),[0,\infty))$ and assume that $0 < M_{q,\alpha} < \infty$. Then $M_{q,\alpha+1} \le M_i \le M_{q,\alpha}$, where M_i is the constant defined by (1.2).

Proof. Let

$$E = \int_0^1 G_{\alpha_i}(s,t)h_i(t)q(t)dt \le ah_i(s), \quad s \in [0,1] : a > 0,$$

where $h_i(s) = s^{\alpha-1}(1-s)$, $s \in [0,1]$.

By inequality (2.1), we obtain

INTERNATIONAL JOURNAL OF PROGRESSIVE RESEARCH IN ENGINEERING MANAGEMENT AND SCIENCE (IJPREMS)
(Int Peer Reviewed Journal)
Vol. 05, Issue 04, April 2025, pp : 2590-2595
e-ISSN : 2583-1062
Impact Factor : 7.001
www.ijprems.com
editor@ijprems.com

$$\int_0^1 G_{\alpha_i}(s,t)h_i(t)q_i(t)dt \leq \frac{1}{\Gamma(\alpha-1)}s^{(\alpha-2)}\int_0^1 t^{\alpha-1}(1-t)^{\alpha-1}MIN(s,t)q_i(t)dt$$

$\leq M_{q,\alpha}h_i(s)$, where $MIN = min(s,t)\Big(1 - max(s,t)\Big)$.

It follows that $E \neq \varphi$ and $M_i \leq M_{q,\alpha}$, where $M_i = \inf E$.

On the other hand, using inequality (2.1) and

$\min(s,t)(1 - \max(s,t)) \geq st(1 - s)(1 - t)$ for $t \in [0,1]$, we deduce that for any $a \in E$,

$$\begin{aligned} ah_i(s) &\geq \frac{1}{\Gamma(\alpha)}s^{\alpha-2}\int_0^1 t^{\alpha-1}(1-t)^{\alpha-1}min(s,t)\Big(1-max(s,t)\Big)q_i(t)dt,\\ &\geq \frac{1}{\Gamma(\alpha)}s^{\alpha-2}\int_0^1 t^{\alpha-1}(1-t)^{\alpha-1}st(1-s)(1-t)q_i(t)dt,\\ &= h_i(s)M_{q,\alpha+1}. \end{aligned}$$

Hence for each $a \in E$, $a \geq M_{q,\alpha+1}$. Therefore $M_i \geq M_{q,\alpha+1}$. Thus the result.

Remark 3.1 From Lemma 3.1, it is obvious that if $M_{q,\alpha} < 1$, then $M_i = \inf E < 1$.

Note that the inequality $M_{q,\alpha} < 1$ can be verified for a large class of functions q, including the singular cases.

Lemma 3.2 If $\alpha \in (2,3)$ and let φ be a function such that $s \to (1 - s)^{\alpha-1}\varphi(s) \in C((0,1)) \cap L^1((0,1))$.

Then the unique continuous solution of the problem

$$D^\alpha v_i(s) = -\varphi(s), \qquad s \in (0,1) \qquad (3.1)$$

$$v_i(0) = v_i^0(0) = v_i(1) = 0$$

is given by

$$V_i\phi(s) = \int_0^1 G_{\alpha_i}(s,t)\phi(t)dt.$$

Proof. Let φ be a function such that $s \to (1-s)^{\alpha-1}\varphi(s) \in C((0,1))\cap L^1((0,1))$. Since by Lemma 2.2, $G_{\alpha i}(s,t)$ belongs to $C([0,1] \times [0,1])$ with $0 \leq G_{\alpha_i}(s,t) \leq \frac{1}{\Gamma(\alpha-1)}(1-t)^{\alpha-1}$,

By the dominated convergence theorem it follows that $V_i\varphi \in C([0,1])$ and $V_i\varphi(0) = V_i\varphi(1) = 0$. Therefore $I^{3-\alpha}(V_i|\varphi|)$ is bounded on [0,1]. By Fubini's theorem, we obtain

$$\begin{aligned} I^{3-\alpha}(V_i\phi)(s) &= \frac{1}{\Gamma(3-\alpha)}\int_0^s (s-t)^{2-\alpha}V_i\phi(t)dt\\ &= \int_0^1 K_i(s,r)\phi(r)dr, \end{aligned}$$

where $K_i(s,r) = \frac{1}{\Gamma(3-\alpha)}\int_0^s (s-t)^{2-\alpha}G_\alpha(t,r)dt.$

Thus $K_i(s,r) = \frac{1}{2}s^2(1-r)^{\alpha-1} - \frac{1}{2}((s-r)^+)^2$.

Hence, for $s \in (0,1)$, we have

$$I^{3-\alpha}(V_i\phi)(s) = \frac{s^2}{2}\int_0^1 (1-r)^{\alpha-1}\phi(r)dr - \frac{1}{2}\int_0^s (s-r)^2\phi(r)dr.$$

This implies that

$$\frac{d^3}{ds^3}\Big(I^{3-\alpha}(V_i\phi)\Big)(s) = -\phi(s).$$

Now, since for each $y \in (0,1)$,

$$\lim_{s\to 0}\frac{G_{\alpha_i}(s,t)}{s} = 0 \quad and \quad 0 \leq \frac{G_{\alpha_i}(s,t)}{s} \leq \frac{1}{\Gamma(\alpha-1)}(1-t)^{\alpha-1}$$

By the dominated convergent theorem, we obtain $(V_i\varphi)^0(0) = 0$.

To prove the uniqueness, let $v_i, w_i \in C([0,1])$ be two solutions of problem (3.1) and set $\theta_i = v_i - w_i$. Then $\theta_i \in C([0,1])$, and we have

$$D^\alpha\theta_i(s) = 0, \qquad s \in (0,1)$$

www.ijprems.com
editor@ijprems.com
INTERNATIONAL JOURNAL OF PROGRESSIVE RESEARCH IN ENGINEERING MANAGEMENT AND SCIENCE (IJPREMS)
(Int Peer Reviewed Journal)
Vol. 05, Issue 04, April 2025, pp : 2590-2595
e-ISSN : 2583-1062
Impact Factor : 7.001

$\theta_i(0) = \theta_i^0(0) = \theta_i(1) = 0$.

By [[9], Corollary 2.1] there exist $c_1, c_2, c_3 \in R$ such that

$\theta i(s) = c_1 s^{\alpha-1} + c_2 s^{\alpha-2} + c_3 s^{\alpha-3}$.

Applying the boundary conditions, we obtain $c_3 = c_2 = c_1 = 0$, that is $v_i = w_i$. This proves the uniqueness of solutions of the problem (3.1).

Remark 3.2 The conclusion of Lemma 3.1 remains true for $\alpha = 3$.

Theorem 3.1 Assume that (A_1) and (A_2) hold. If $M < 1$, then problem (1.1) has a unique solution $v_i \in C([0,1])$. In addition, for any $v_0 \in C([0,1])$, the iterative sequence

$$v_{k_i}(s) = \int_0^1 G_{\alpha_i}(s,t) f_i(t, v_{k_i-1}, .)dt$$

converging to v_i with respect to the h−norm, and we have

$$\|v_{k_i} - v_i\| \le \frac{M^k}{1-M}\|v_1 - v_0\| \qquad (3.2)$$

Proof. Define the operator T by

$$Tv_i(s) = \int_0^1 G_{\alpha_i}(s,t) f_i\Big(t, v_i(t)\Big)dt, \quad s \in [0,1], \quad v_i \in C\Big([0,1]\Big). \qquad (3.3)$$

We claim that T is a contraction operator from $\Big(C([0,1]), \|.\|\Big)$ into itself.

Let $v_i \in C\Big([0,1]\Big)$, and let $\sigma > 0$ be such that $|v_i(s)| \le \sigma h_i(s)$ for all $s \in [0,1]$. By Lemma 2.2 (ii), and

$$0 \le G_{\alpha_i}(s,t) \le \frac{1}{\Gamma(\alpha-1)}(1-t)^{\alpha-2},$$

it follows from (A_2) that

$$\begin{aligned}
\Big|G_{\alpha_i}(s,t) f_i\big(t, v_i(t)\big)\Big| &\le \frac{1}{\Gamma(\alpha-1)}(1-t)^{\alpha-2}\Big(\big|f_i\big(t, v_i(t)\big) - f_i(t,0)\big| + \big|f_i(t,0)\big|\Big) \\
&\le \frac{1}{\Gamma(\alpha-1)}(1-t)^{\alpha-2}\Big(q_i(t)|v_i(t)| + |f_i(t,0)|\Big) \\
&\le \frac{1}{\Gamma(\alpha-1)}\Big(\sigma t^{\alpha-1}(1-t)^{\alpha-1}q_i(t) + (1-t)^{\alpha-2}|f_i(t,0)|\Big)
\end{aligned}$$

Since $G_{\alpha i}(s,t)$ is continuous on $[0,1] \times [0,1]$, by (A_1)-(A_2) and the dominated convergence theorem we deduce that $Tv_i \in C([0,1])$.

Furthermore, from Lemma 2.2 (ii), we have

$$0 \le G_{\alpha_i}(s,t) \le \frac{1}{\Gamma(\alpha-1)} h_i(x)(1-t)^{\alpha-2}, \qquad (3.4)$$

Hence by using inequality (3.4) and similar arguments as before we obtain

$$\big|Tv_i(s)\big| \le \left[\sigma M_{q,\alpha} + \frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-t)^{\alpha-2}\big|f_i(t,0)\big|\right] h_i(t)$$

and thus $T(C([0,1])) \subset C([0,1])$.

Next, for any $v_i, w_i \in C([0,1])$, using (A_2), we obtain for $s \in [0,1]$,

$$\begin{aligned}
\big|Tv_i(s) - Tw_i(s)\big| &\le \int_0^1 G_{\alpha_i}(s,t)\big|f_i(t, v(t)) - f_i(t, w(t))\big|dt \\
&\le \int_0^1 G_{\alpha_i}(s,t) q_i(t)\big|v_i(t) - w_i(t)\big|dt \\
&\le \|v_i - w_i\| \int_0^1 G_{\alpha_i}(s,t) q_i(t) h_i(t)dt \\
&\le M\|v_i - w_i\| h_i(s).
\end{aligned}$$

Hence $\|Tv_i - Tw_i\| \leq M\|v_i - w_i\|h_i(s)$. Since $M < 1$, T becomes a contraction operator in C([0,1]). So there exists a unique $v_i \in C([0,1])$ satisfying

$$v_i(x) = \int_0^1 G_{\alpha_i}(s,t) f_i(t, v_i(t))dt, \quad s \in (0,1).$$

It remains to prove that v_i is a solution of problem (1.1). It is clear that

$s \to (1-s)^{\alpha-1} f_i(s, v_i(s)) \in C((0,1))$.

Next, by using (A_2) and $v_i \in C([0,1])$ we obtain

$$\begin{aligned}\left|(1-s)^{\alpha-1} f_i(s, v_i(s))\right| &\leq (1-s)^{\alpha-1}\left|\left(f_i(s,v_i(s)) - f_i(s,0)\right| + \left|f_i(s,0)\right|\right.\\ &\leq (1-s)^{\alpha-1}\left(q_i(s)\left|v_i(s)\right| + \left|f_i(s,0)\right|\right)\\ &\leq \sigma s^{\alpha-1}(1-s)^{\alpha-1} q_i(s) + (1-s)^{\alpha-2}\left|f_i(s,0)\right|.\end{aligned}$$

Therefore by (A_1) and (A_2), it follows that $s \to (1-s)^{\alpha-1} f_i(s, v_i(s)) \in L^1((0,1))$.

Hence from Lemma 3.1, we derive that v_i is a solution of problem (1.1).

Finally, it is known that for any $v_i \in C([0,1])$, the iterative sequence

$$v_{k_i}(s) = \int_0^1 G_{\alpha_i}(s,t) f_i\left(t, v_{k_i-1}(t)\right)dt$$

converges to v_i, and we have

$$\|v_{k_i} - v_i\| \leq \frac{M^k}{1-M}\|v_1 - v_0\|.$$

This proves the theorem.

2. CONCLUSION

The existence and uniqueness of solution for system of fractional nonlinear boundary value problem (1.1) is obtained using Green function. It is also proved that the obtained sequence of iterations of solutions $v_{ki}(s)$ converges to v_i.

3. REFERENCES

[1] Atangana A., Akgul A. Analysis of new trends of fractional differential equations. In : Fractional Order Analysis, Wiley, USA(2020), 91-111.

[2] Bachar I., Maagli H.,Eltayeb H. Existence and uniqueness of solutions for a calss of fractional nonlinear boundary value problems under mild assumptions, Advances in difference Equations, Vol (2021) 2021:22.

[3] Bai Z. On On positive solutions of a nonlocal fractional boundary value problem. Nonlinear Anal. 72(2), (2010) 916-924.

[4] Bai Z., Zhang S. Sun S., Yin,C. Mnotone iterative method for fractional differential equations. Electron. J. Differ. Equ. 2016, 6(2016).

[5] Cui Y. Uniqueness of solution for boundary value problems for fractional differential equations. Appl.Math.Lett.51, (2016) 48-54.

[6] Chatibi Y., Kinani E.H.E., Ouhadan A., Lie Symmetry analysis of conformable differential equaitons. AIMS Math. 4, (2019) 1133-1144.

[7] Diethelm K. The analysis of Fractional Differential Equations. Lecture Notes in Mathematics, Vol.2004. Springer, Berlin (2010).

[8] Dhaigude D.B., Nanware J.A., Nikam V.R. Monotone technique for system of caputo fractional differential equations with periodic boundary conditions. Dynamics of Continuous, Discrete and Impulsive System Series A : Mathematical Analysis 19, (2012) 575-584.

[9] Kilbas A.A., Srivastava H.M., Trujillo J.J. Theory and Applications of Fractional Differential Equations. North-Holland Mathematrics Studies, Vol.204. Elsevier, Amsterdam (2006).

[10] Liang S., Zhang J. Positive solutions for boundary value problems of nonlinear fractional differential equation. Nonlinear Anal.71(11), (2009) 5545-5550.

[11] Nanware J.A., Jadhav N.B., Dhaigude D.B. Existence of uniqueness of solutions of nonlinear implicit fractional differential equation. Dynamics of Continuous, Discrete and Impulsive System Series A : Mathematical Analysis 27, (2020) 275-282.

[12] Podlubny I. Fractional differential equations Academic Press. 198, San Diego,(1999).

[13] Starrett J. Solving differential equations by symmetry groups. Mar. Math. Mon. 114, (2007) 778-792.

[14] Zhang X., Liu L., Wu Y. Multiple positive solutions of a singular fractional differetial equation with negatively perturbed term. Math. Comput, Model. 55(34), (2012) 1263-1274.

[15] Zhang X., Liu L., Wu Y. The uniqueness of positive solution for a fraztional order model of turbulent flow in porous medium. Appl.Math.Lett.37, (2014) 26-33.

[16] Zou Y., He G. On the uniqueness of solutions for a class of fractional differential equations. Appl. Math. Lett. 74, (2017) 68-73.