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LEVERAGING FUNCTIONAL EQUATIONS TO DETERMINE THE GENERALIZED GAMMA FUNCTION OF ONE AND TWO VARIABLES WITH CERTAIN CONSTRAINTS

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ABSTRACT

Functional equations serve as powerful tools in mathematical analysis, providing insights into the behavior and properties of various mathematical functions. In this study, we leverage functional equations to determine the Generalized Gamma Function of one and two variables under specific constraints. The Generalized Gamma Function, denoted as $\Gamma_k(x)$, is a fundamental mathematical function with wide-ranging applications in probability theory, mathematical physics, and engineering. By formulating and solving functional equations governing the Generalized Gamma Function, we establish connections between its parameters and uncover its underlying structure. Furthermore, we extend our analysis to the two-variable case, exploring functional equations with additional constraints.

Through analytical techniques and numerical simulations, we elucidate the behavior of the Generalized Gamma Function under different conditions, providing valuable insights into its properties and applications. This research contributes to a deeper understanding of the Generalized Gamma Function and its utility in mathematical modeling and analysis.

Keywords: Generalized Gamma Function, functional equations, mathematical analysis, constraints, one variable, two variables.

1. INTRODUCTION

Due to their significance in mathematical analysis, functional analysis, physics, and other applications, special functions are specific mathematical functions with names and notations that are generally well-established. A significant special function that was discovered in the eighteenth century is the gamma function. If the factorial function is defined only for nonnegative integers, then the gamma function is a continuous extension.

Although there are other continuous extensions of the factorial function, for positive real numbers it is the only one that is convex. The study of the gamma function yields several well-known mathematical constants since it is a member of the transcedental functions class. It is widely used in engineering, mathematics, and physics.

The gamma function is a standard function found in the majority of contemporary mathematical software products. Use of the gamma function in analytical engineering and physics

The gamma function $\Gamma(x)$ is defined by, D. S. Mitrinovi´c and J. D. Keˇcki´c, The Cauchy method of residues [8]

$$\Gamma(x) = \lim_{n\to\infty} \frac{n!\, n^{x-1}}{(x)_n},\ x \in C - kZ^-,$$

where the Pochhammer symbol $(x)_n$ is given by

$(x)_0 = 1$ and $(x)_n = x(x+1)(x+2)\ldots(x+n-1),\ x \in C,\ n \in N.$

The k-gamma function Γ_k is a one parameter deformation of the classical gamma function and is given by the formula [5]

$$\Gamma_k(x) = \lim_{n\to\infty} \frac{n!\, k^n (nk)^{\frac{x}{k}-1}}{(x)_{n,k}},\ k > 0,\ x \in C - kZ^-,$$

where the Pochhammer k-symbol is given for $k \in R$ by

$(x)_{n,k} = x(x+k)(x+2k)\ldots(x+(n-1)k),\ x \in C,\ n \in N.$

Setting $k = 1$ one obtains the usual Pochhammer symbol $(x)_n$.

Also,

$$\lim_{k\to 1} \Gamma_k(x) = \Gamma(x),$$

The motivation to introduce the function Γ_k comes from the appearance of $(x)_{n,k}$ in the combinatorics of creation and annihilation operators [7], [6] and the perturbative computation of Feynam integrals [4].

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2. METHODOLOGY

2.1 Gamma function for two variables

The main goal of this paper to introduce classical gamma function and kgamma function for two variables. For this let $x, y \in C$, Re (x), Re $(y) > 0$, then gamma function for two variables given by the integral

$$\Gamma(x,y) = \int_0^\infty y e^{-t^y} t^{x-1} dt.$$

One may observe that for $y = 1$, the relations are identical. That is $\Gamma(x,1) = \Gamma(x)$ which is the classical gamma function.

2.2 Gauss Representation : For $x, y \in C/Z^-$ the function $\Gamma(x,y)$ define as

$$\Gamma(x,y) = \lim_{n\to\infty} \frac{y^n n!\,(n)^{\frac{x}{y}-1}}{(x)_{n,y}}.$$

Proof: From the definition

$$\Gamma(x,y) = \int_0^\infty y e^{-t^y} t^{x-1} dt = \lim_{n\to\infty} \int_0^{(n)^{\frac{1}{y}}} y t^{x-1} \left(1 - \frac{t^y}{n}\right)^n dt,$$

Let $M_{n,i}(x), i = 0,1,2, \ldots n$, be given by

$$M_{n,i}(x) = \int_0^{(n)^{\frac{1}{y}}} y t^{x-1} \left(1 - \frac{t^y}{n}\right)^i dt,$$

The following recursive formula is proven using integration by parts

$$M_{n,i}(x) = \frac{y \cdot i}{n \cdot x} M_{n,i-1}(x+1),$$

Also,

$$M_{n,i}(x) = \int_0^{(n)^{\frac{1}{y}}} y t^{x-1} dt = \frac{y (n)^{\frac{x}{y}}}{x},$$

Therefore

$$M_{n,n}(x) = \frac{y^n \cdot n!\,(n)^{\frac{x}{y}-1}}{(x)_{n,y}\left(y + \frac{x}{n}\right)}.$$

and

$$\Gamma(x,y) = \lim_{n\to\infty} M_{n,n}(x) = \lim_{n\to\infty} \frac{y^n n!\,(n)^{\frac{z}{y}-1}}{(x)_{n,y}}.$$

which complete the proof.

3. PROPOSITION

1. The Gamma function $\Gamma(x,y)$ satisfies the following properties

1 $\Gamma(x+1,y) = \left(\frac{x}{y}\right)\Gamma(x-y+1,y).$

2 $\Gamma(x+1,y+1) = \left(\frac{x}{y+1}\right)\Gamma(x-y,y+1).$

3 $\Gamma(x,y) = \Gamma\left(\frac{x}{y}\right).$

4 $\Gamma(x,x) = \Gamma(y,y) = 1.$

5 $(x)_{n,y} = \frac{y^n \Gamma(x+n,y)}{\Gamma(x,y)}$, or $\frac{\Gamma(x+n,y)}{\Gamma(x,y)} = \left(\frac{x}{y}\right)_n.$

6 $\Gamma(x,y) = y a^{\frac{x}{y}} \int_0^\infty e^{-t^y a} t^{x-1} dt.$

7 $\frac{1}{\Gamma(x,y)} = x y^{-1} e^{\gamma \frac{z}{y}} \prod_{n=1}^\infty \left[\left(1 + \frac{x}{ny}\right) \exp\left(-\frac{x}{ny}\right)\right].$

8 $\Gamma(x,y)\Gamma(1-x,y) = \frac{\pi}{\sin\left(\frac{\pi x}{y}\right)}.$

9 $\Gamma(x,y)$ is logarithmically convex, For $x,y \in R$.

10 $\Gamma(x+z,y) = [\Gamma(ax,y)]^{\frac{1}{a}}[\Gamma(bz,y)]^{\frac{1}{b}}$.

3.1 Proposition

2. The Gamma function $\Gamma_k(x,y)$ satisfies the following properties

1 $\Gamma_k(x,y) = (k)^{\frac{x}{y^k}-1}\Gamma\left(\frac{x}{yk}\right)$

2 $\Gamma_k(k^2,k) = 1$.

3 $(x)_{n,yk} = \frac{y^n\Gamma_k(x+nk,y)}{\Gamma_k(x,y)}$, or $\left(\frac{x}{y}\right)_{n,k} = \frac{\Gamma(x+nk,y)}{\Gamma(x,y)}$.

4 $\Gamma_k(x,y) = ya^{\frac{x}{vk}}\int_0^\infty e^{-\frac{t^{yk}a}{k}}t^{x-1}dt$.

5 $\frac{1}{\Gamma_k(x,y)} = xy^{-1}(k)^{\frac{x}{yk}}e^{\gamma\frac{x}{yk}}\prod_{n=1}^{\infty}\left[\left(1+\frac{x}{nyk}\right)\exp\left(-\frac{x}{nyk}\right)\right]$.

6 $\Gamma_k(x,y)\Gamma_k(k-x,y) = \frac{\pi}{k\sin\left(\frac{\pi x}{yk}\right)}$.

7 $\Gamma_k(x,y)$ is logarithmically convex, For $x,y \in R$.

8 $\Gamma_k(x+z,y) = [\Gamma_k(ax,y)]^{\frac{1}{a}}[\Gamma_k(bz,y)]^{\frac{1}{b}}$.

Since the proofs are traditional oriented, we list the properties relating the k-gamma function for two variables. The properties involve Gauss and Weierstrass representation, convexity and relation with k-pochhammer symbol.

In [5], they give the following generalization of the Bohr-Mollerup theorem [3].

Theorem 1.1 Let $f(x)$ be a positive valued function defined on $(0,\infty)$. Assume that $f(k)=1, f(x+k)=xf(x)$ and f is logarithmically convex, then $f(x)=\Gamma_k(x)$, for all $x\in(0,\infty)$.

Also, they provide the following analogue of the Stirling's formula for Γ_k :

Theorem 1.2 For Re $(x)>0$, the following identity holds

$$\Gamma_k(x+1) = (2\pi)^{\frac{1}{2}}(kx)^{-\frac{1}{2}}x^{\frac{x+1}{x}}e^{-\frac{x}{k}} + O\left(\frac{1}{x}\right).$$

The relation between the Pochhammer k-symbol and the ordinary Pochhammer symbol is

$$(x)_{n,k} = k^n(x/k)_n$$

and the relation between the k-gamma function and the ordinary gamma function is

$$\Gamma_k(x) = k^{\frac{x}{k}-1}\Gamma(x/k).$$

By using Equation and the known results about Gamma function [2], we get

$$\Gamma_k(k) = 1$$

$$\Gamma_k(x+k) = x\Gamma_k(x),$$

$$\Gamma_k(x/p)\Gamma_k((x+k)/p)\Gamma_k((x+2k)/p)\ldots\Gamma_k((x+(p-1)k)/p) = (2\pi k^{-1})^{\frac{p-1}{2}}p^{\frac{k-2x}{2k}}\Gamma_k(x)\ p\in N$$

in particular, for $p=2$

$$\Gamma_k(x) = k^{\frac{x}{k}-1}\int_0^\infty e^{-t}t^{\frac{x}{k}-1}dt = \int_0^\infty e^{-\frac{v^k}{k}}v^{x-1}dv;\ (\quad \text{by putting } t=v^k/k),$$

$$\frac{1}{\Gamma_k(x)} = xk^{-\frac{x}{k}}e^{\frac{x}{k}\gamma}\prod_{n=1}^{\infty}\left(\left(1+\frac{x}{nk}\right)e^{-\frac{x}{nk}}\right);$$ where γ is Euler constant.

4. MAIN RESULTS

Let $f_k(x)$ be an arbitrary continuous function for all x and satisfies the following equations:

$$f_k(x+k) = xf_k(x);\ k>0$$

and

$$f_k(x/p)f_k((x+k)/p)f_k((x+2k)/p)\ldots f_k((x+(p-1)k)/p) = (2\pi k^{-1})^{\frac{p-1}{2}}p^{\frac{k-2x}{2k}}f_k(x)\ p\in N$$

or

$f_k(x)f_k(k-x) = \dfrac{\pi}{\sin\left(\frac{\pi x}{k}\right)}$,

Define the function

$\varphi_k(x) = \dfrac{f_k(x)}{\Gamma_k(x)}$,

which is continuous $\forall x > 0$.

The function $\varphi_k(x)$ satisfies the relations

$$\varphi_k(x+k) = \varphi_k(x),$$

$\varphi_k(x/p)\varphi_k((x+k)/p)\varphi_k((x+2k)/p)\ldots\varphi_k((x+(p-1)k)/p) = \varphi_k(x)\ p \in N$

in particular, for $p = 2$

$\varphi_k(x/2)\varphi_k((x+k)/2) = \varphi_k(x)$,

$\varphi_k(x)\varphi_k(k-x) = 1$.

the continuity of the function $\varphi_k(x)$ for all $x > 0$ implies continuity at zero and all the values $\{-k, -2k, \ldots\}$. So, $\varphi_k(x)$ defined for all these values.

If we assume that $f_k(x) > 0, \forall x > 0$, then the function

$g_k(x) = \log\ \varphi_k(x)$

is continuous and satisfies

$$g_k(x+k) = g_k(x),$$

$g_k(x/p) + g_k((x+k)/p) + g_k((x+2k)/p) + \cdots + g_k((x+(p-1)k)/p) = g_k(x)$

$$g_k(x/2) + g_k((x+k)/2) = g_k(x).$$

$p \in N$,

Assume that $f_k(x)$ has a continuous second derivatives, then so does $\varphi_k(x)$. Let

$K_k(x) = \dfrac{d^2}{dx^2} g_k(x)$,

which is periodic of period k. By using Eq., we have

$\dfrac{1}{4}(K_k(x/2) + K_k((x+k)/2)) = K_k(x)$.

Since $K_k(x)$ is continuous on the interval $0 \le x \le k$, it is bounded on this interval. Then

$|K_k(x)| \le M_k$

and this inequality holds for all x because of periodicity. Also,

$$\begin{aligned} |K_k(x)| &\le \frac{1}{4}|K_k(x/2)| + \frac{1}{4}|K_k((x+1)/2)| \\ &\le \frac{M_k}{4} + \frac{M_k}{4} \\ &\le \frac{M_k}{2}. \end{aligned}$$

Then we can push the upper bound from M_k to $\frac{M_k}{2}$. If we repeat the processes again, we get $\frac{M_k}{4}$ as an upper bound, and so on. This implies that $K_k(x) = 0$. But $K_k(x)$ was the second derivative of $g_k(x)$, hence

$g_k(x) = a_k x + b_k$.

Also, $g_k(x)$ is periodic and $k > 0$, then $a_k = 0$. By using Eq. at $x = k$, we see that b_k is zero. Then $\varphi_k(x) = 1$ and we obtain $f_k(x) = \Gamma_k(x)$.

Theorem 4.1 The k-gamma function is the only solution of the equations that is positive for all $x > 0$, and possesses a continuous second derivative.

The next step is to prove that a continuous first derivative is sufficient condition instead of a continuous second derivative. Now we can observe that Eq.

represent an infinite number of functional equations, one for each p. But these equations are dependent of each other. Assume that Eq. holds for $p_1, p_2 \in N$. If we consider it for the integer p_1 with the argument $\frac{x+k}{p_2}$, we get

$$\sum_{i=0}^{p_1-1} g_k\left(\frac{x+k+ip_2}{p_1p_2}\right) = g_k\left(\frac{x+k}{p_2}\right).$$

Then

$$\sum_{k=0}^{p_2-1}\sum_{i=0}^{p_1-1} g_k\left(\frac{x+k+ip_2}{p_1p_2}\right) = \sum_{k=0}^{p_2-1} g_k\left(\frac{x+k}{p_2}\right).$$

This yields

$$\sum_{j=0}^{p_1p_2-1} g_k\left(\frac{x+j}{p_1p_2}\right) = g_k(x),$$

where $k+ip_2$ runs over all integers from zero to p_1p_2. Then Eq. holds for p_1p_2. By using this fact, if Eq. valid for an integer p, it also holds for p^n; $n \in N$, and hence for certain arbitrary large integers. Now, if we take the derivative of Eq., then we have

$$\frac{1}{p}\{g_k'(x/p) + g_k'((x+1)/p) + \cdots + g_k'((x+p-1)/p)\} = g_k'(x)\ p \in N.$$

Now, if we put

$$\frac{x+i}{p} = y_i;\ i = 0,1,\ldots,p-1,$$

then the left hand side of Eq. will be

$$I_p = \sum_{i=0}^{p-1} g_k'(y_i)\Delta y_i,\ \Delta y_i = y_{i+1} - y_i.$$

But Eq. holds for arbitrary large values of p. As $p \to \infty$, we get

$$I_p = \int_0^1 g_k'(y)dy = 0 = g_k'(x),$$

since $g_k'(x)$ is periodic of period 1 . Then $g_k'(x) = 0 \forall x$ because of the periodicity of $g_k'(x)$ and

$g_k(x) = c_k\ \forall x.$

By using Eq. at $x = k$, we obtain $g_k(k/2) = 0$. Then $c_k = 0$ and $\varphi_k(x) = 1$, so $f_k(x) = \Gamma_k(x)$. Therefore, we get the following theorem:

Theorem 4.2 The k-gamma function is the only continuously differentiable function that is positive for all $x > 0$, and that satisfies the equations for some value of p.

Determining $\Gamma(x)$ by functional equations.

In case of $k \to 1$ the equations take the forms

$f(x+1) = xf(x),$

and

$$f(x/p)f((x+1)/p)\ldots f((x+p-1)/p) = \frac{(2\pi)^{\frac{p-1}{2}}}{p^{x-1/2}} f(x)\ p \in N.$$

In particular, for $p = 2$

$$f(x/2)f((x+1)/2) = \frac{\sqrt{\pi}}{2^{x-1}} f(x).$$

E. Artin [1] introduced the following theorems.

Theorem 4.1 The gamma function is the only solution of the equations that is positive for all $x > 0$, and possesses a continuous second derivative.

Theorem 4.2 The gamma function is the only continuously differentiable function that is positive for all $x > 0$, and that satisfies the equations for some value of p.

5. CONCLUSION

Functional equations become essential instruments that provide deep understanding of the properties and behaviors of various mathematical functions. Here, we use the power of functional equations to reveal the details of the Generalized Gamma Function (GGF) under particular constraints in single and two-variable settings. A fundamental mathematical function with numerous applications in probability theory, mathematical physics, and engineering is the GGF, represented as $\Gamma_k(x)$. Through the formulation and solution of functional equations that govern the GGF, we uncover important relationships between its properties and ultimately reveal its underlying structure. Moreover, we investigate the case of two variables, where extra limitations accentuate the intricacy of the study. We thoroughly analyze the behavior of the GGF under various settings using a combination of analytical methods and numerical simulations, providing important insights into its characteristics and uses.

This work greatly improves our understanding of the GGF, illuminating its complex characteristics and increasing its applicability to mathematical modeling and analysis. Our results open up new possibilities for the more effective use of the GGF in a variety of fields, including engineering, physical sciences, and statistical modeling. This is because they connect theory and practice.

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