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PATTERNS OF INTEGER SOLUTIONS TO NON-HOMOGENEOUS TERNARY CUBIC EQUATION $2(x^2+y^2)-3xy=14z^3$

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ABSTRACT

This paper concentrates on illustrating varieties of integer solutions to non-homogeneous ternary cubic Diophantine equation $2(x^2+y^2)-3xy=14z^3$.

Keywords: Non-homogeneous cubic ,Ternary cubic ,Integer solutions, Substitution technique , Factorization method

1. INTRODUCTION

The theory of Diophantine equations is an ancient subject that typically involves solving, polynomial equation in two or more variables or a system of polynomial equations with the number of unknowns greater than the number of equations, in integers and occupies a pivotal role in the region of mathematics.The subject of Diophantine equations has fascinated and inspired both amateurs and mathematicians alike and so they merit special recognition. Solving higher degree diophantine equations can be challenging as they involve finding integer solutions that satisfy the given polynomial equation. Learning about the various techniques to solve these higher power diophantine equation in successfully deriving their solutions help us understand how numbers work and their significance in different areas of mathematics and science. For the sake of clear understanding by the readers, one may refer the varieties of cubic Diophantine equations with multi variables [1-18]. This paper aims at determining many integer solutions to non-homogeneous polynomial equation of degree three with three unknowns given by $2(x^2+y^2)-3xy=14z^3$.

Technical procedure

The non-homogeneous ternary cubic equation is

$$2(x^2+y^2)-3xy=14z^3 \tag{1}$$

By inspection , it is observed that (1) is satisfied by the integer triples

$(x,y,z)=(14^2\alpha^{3s},14^2\alpha^{3s},14\alpha^{2s}),(-4\alpha^{3s},4\alpha^{3s},2\alpha^{2s})$. However , there are many more choices of integer solutions to (1). The process of getting the same is illustrated below:

Choice 1

The substitution

$$x=ky,\ k\neq\pm1 \tag{2}$$

in (1) gives

$$(2k^2-3k+2)\,y^2=14z^3 \tag{3}$$

which is satisfied by

$$y=14^2(2k^2-3k+2)\,\alpha^{3s},$$
$$z=14\,(2k^2-3k+2)\,\alpha^{2s}. \tag{4}$$

From (2) ,we get

$$x=14^2k(2k^2-3k+2)\,\alpha^{3s} \tag{5}$$

Thus, (4) & (5) satisfy (1).

Choice 2

The option

$$x=(k+1)v,\ y=(k-1)v,\ k\neq\pm1 \tag{6}$$

in (1) gives

$$(k^2+7)\,v^2=14z^3$$

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which is satisfied by

$$v = 14^2 (k^2 + 7)\, \alpha^{3s} \tag{7}$$

and

$$z = 14(k^2 + 7)\, \alpha^{2s} \tag{8}$$

Using (7) in (6) , we have

$$\begin{aligned} x &= 14^2 (k+1)\, (k^2 + 7)\, \alpha^{3s}, \\ y &= 14^2 (k-1)\, (k^2 + 7)\, \alpha^{3s}. \end{aligned} \tag{9}$$

Thus,(8) & (9) satisfy (1).

Choice 3

The option

$$x = (k+1)\, v,\; y = (1-k)\, v,\; k \neq \pm 1 \tag{10}$$

in (1) gives

$$(1 + 7k^2)\, u^2 = 14 z^3$$

which is satisfied by

$$u = 14^2 (1 + 7k^2)\, \alpha^{3s} \tag{11}$$

and

$$z = 14(1 + 7k^2)\, \alpha^{2s} \tag{12}$$

Using (11) in (10) , we have

$$\begin{aligned} x &= 14^2 (1+k)\, (k^2 + 7)\, \alpha^{3s}, \\ y &= 14^2 (1-k)\, (k^2 + 7)\, \alpha^{3s}. \end{aligned} \tag{13}$$

Thus,(12) & (13) satisfy (1).

Choice 4

The substitution

$$x = u + v,\; y = u - v,\; u \neq v \neq 0 \tag{14}$$

in (1) leads to

$$u^2 + 7v^2 = 14 z^3 \tag{15}$$

Assume

$$z = a^2 + 7b^2 \tag{16}$$

Consider the integer 14 on the R.H.S. of (15) as

$$14 = \frac{(7 + i\sqrt{7})\,(7 - i\sqrt{7})}{4} \tag{17}$$

Substituting (16) & (17) in (15) and applying factorization , consider

$$u + i\sqrt{7}\, v = \frac{(7 + i\sqrt{7})}{2} (a + i\sqrt{7}\, b)^3$$

On comparing the coefficients of corresponding terms ,we have

$$u = \frac{7}{2}[f(a,b) - g(a,b)]\,,$$

$$v = \frac{1}{2}[f(a,b) + 7\,g(a,b)].$$

where

$$f(a,b) = a^3 - 21ab^2,$$
$$g(a,b) = 3a^2b - 7b^3.$$

In view of (14) ,we get

$$x = 4f(a,b),$$
$$y = 3f(a,b) - 7g(a,b). \qquad (18)$$

Thus, (16) & (18) satisfy (1) .

Note 1

Apart from (17) ,the integer 14 may also be represented as shown below:

$$14 = \frac{(7+i5\sqrt{7})\,(7-i5\sqrt{7})}{16}$$
$$14 = \frac{(7+i11\sqrt{7})\,(7-i11\sqrt{7})}{64}$$
$$14 = \frac{(7+i31\sqrt{7})\,(7-i31\sqrt{7})}{484}$$

Following the above procedure, three more sets of integer solutions to (1) are obtained.

Choice 5

Substitution of

$$x = u + z,\ y = u - z \qquad (19)$$ in (1) gives

$$u^2 = z^2(14z - 7) \qquad (20)$$

After performing some algebra ,it is seen that $(14z-7)$ is a perfect square

when

$$z = 14s^2 + 14s + 4 \qquad (21)$$

and from (20) ,we get

$$u = (14s+7)\,(14s^2+14s+4)$$

In view of (19) ,we have

$$x = (14s^2+14s+4)\,(14s+8),$$
$$y = (14s^2+14s+4)\,(14s+6). \qquad (22)$$

Thus , (21) & (22) satisfy (1).

Note 2

It is to be noted that $(14z-7)$ in (20) is also a perfect square when

$$z = 14s^2 - 14s + 4$$

In this case , the values of x, y satisfying (1) are given by

$$x = (14s^2-14s+4)\,(14s-6),$$
$$y = (14s^2-14s+4)\,(14s-8).$$

Choice 6

Write (15) as

$$u^2 + 7v^2 = 14z^3 * 1 \qquad (23)$$

Express the integer 1 as

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$$1=\frac{(3+i\sqrt{7})\,(3-i\sqrt{7})}{16} \qquad (24)$$

Substituting (16) ,(17) & (24) in (23) and using factorization ,consider

$$(u+i\sqrt{7}\,v)=\frac{(7+i\sqrt{7})}{2}\frac{(3+i\sqrt{7})}{4}(a+i\sqrt{7}\,b)^3$$

Following the procedure as in Choice 4 , the corresponding integer solutions to (1) are found to be

$$x=8(3f(a,b)-7g(a,b))\,,$$
$$y=4\,(f(a,b)-21g(a,b))\,,$$
$$z=4(a^2+7b^2)\,.$$

Note 3

It is worth to mention that the integer 1 in (23) may be represented as below:

$$1=\frac{(1+i3\sqrt{7})\,(1-i3\sqrt{7})}{64}\,,$$

$$1=\frac{(7r^2-s^2+i\,\sqrt{7}\,2rs)\,(7r^2-s^2-i\,\sqrt{7}\,2rs)}{(7r^2+s^2)^2}\,,$$

$$1=\frac{(r^2-7s^2+i\,\sqrt{7}\,2rs)\,(r^2-7s^2-i\,\sqrt{7}\,2rs)}{(r^2+7s^2)^2}.$$

The repetition of the above process gives three more sets of integer solutions to (1).

Choice 7

The substitution of the transformations

$$x=7kz+v\,,y=7kz-v \qquad (25)$$

in (1) leads to

$$v^2=z^2(2z-7k^2) \qquad (26)$$

The R.H.S. of (26) is a perfect square when

$$z=z_0=(2s^2+2s+4)\,k^2 \qquad (27)$$

From (26) , we have

$$v=v_0=(2s+1)\,(2s^2+2s+4)k^3 \qquad (28)$$

Employing (27) & (28) in (25) ,it is seen that

$$x=k^3(2s^2+2s+4)\,(2s+8)\,,$$
$$y=k^3(2s^2+2s+4)(6-2s)\,. \qquad (29)$$

Thus , (27) & (29) satisfy (1) .

Note 4

The R.H.S. of (26) is also a perfect square for values of z given by

$$z_n=z_0+2n^2+2\,n(2s+1)k \qquad (30)$$

From (26) , we get

$$v_n=z_n[(2s+1)k+2n] \qquad (31)$$

In view of (25) , we have

$$x_n=z_n\,(2n+(2s+8)k)\,,$$
$$y_n=z_n.(-2n+(-2s+6)k) \qquad (32)$$

Thus , (30) & (32) satisfy (1) .

Note 5

It is worth mentioning that , in addition to (27) & (28) , one may have

$$z = z_0 = (2s^2 - 2s + 4)\, k^2$$

$$v = v_0 = (2s-1)\,(2s^2 - 2s + 4)\, k^3$$

The repetition of the above process gives a different set of integer solutions to (1).

2. CONCLUSION

In this paper , varieties of integer solutions to non-homogeneous ternary cubic equation given in title are obtained. As cubic equations are plenty , one may search for integer solutions to other forms of cubic Diophantine equations.

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